Fractional Diffusion – Theory and Applications – Part I

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The mathematical description of diffusion has many formulations:

- conservation of mass and constitutive laws
- probabilistic models based on random walks and central limit theorems
- microscopic stochastic models based on Brownian motion and Langevin equations
- mesoscopic stochastic models based on master equations and Fokker-Planck equations
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The mean square displacement \( \langle (X - \langle X \rangle)^2 \rangle \) of a diffusing particle scales linearly with time.
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Numerous experimental measurements in spatially complex systems have revealed anomalous diffusion in which the mean square displacement scales as a fractional order power law in time.

Over the past two decades a new mathematical description has been formulated, linked together by tools of fractional calculus:

- fractional constitutive laws
- probabilistic models based on continuous time random walks and generalized central limit theorems
- fractional Langevin equations, fractional Brownian motions
- fractional diffusion, fractional Fokker-Planck equations
A Triptych Overview
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- Brownian motion
- Standard random walks
- Standard diffusion
A Triptych Overview

- Brownian motion
- standard random walks
- standard diffusion
- fractional calculus
A Triptych Overview

- Brownian motion
  - standard random walks
  - standard diffusion
- fractional calculus
- fractional Brownian motion
  - continuous time random walks
  - fractional diffusion
Having found motion in the particles of the pollen of all the living plants which I had examined, I was led next to inquire whether this property continued after the death of the plant, and for what length of time it was retained.

Robert Brown (1828)
Einstein’s Relations (1905)

It is possible that the movements to be discussed here are identical with the so-called “Brownian molecular motion” ; however, the information available to me regarding the latter is so lacking in precision, that I can form no judgment in the matter.
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\[ \langle r^2(t) \rangle = 2Dt \quad D = \left( \frac{RT}{6N\pi a\eta} \right) = \left( \frac{k_BT}{\gamma} \right) \]

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Einstein Relations

\[ \gamma = 6\pi \eta a \text{ (Stokes)}, \quad T \text{ temperature of fluid}, \quad R = Nk_B \text{ gas constant}, \]

\[ a \text{ particle radius}, \quad \eta \text{ fluid viscosity}, \quad N \text{ Avogadro’s number} \]
Langevin’s Equation (1908)

\[ m \frac{d^2 x}{dt^2} = F(t) - \gamma \frac{dx}{dt} \]

random force \hspace{2cm} \text{viscous drag}
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\[ \langle F(t) \rangle = 0, \quad \langle F(0)F(t) \rangle = D\delta(t) \quad \langle x(t)F(t) \rangle = \langle x(t) \rangle \langle F(t) \rangle = 0 \]
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\[ \frac{1}{2} m \left\langle \left( \frac{dx}{dt} \right)^2 \right\rangle = \frac{1}{2} k_B T \]

Equipartition of energy (1D) – Boltzmann (1896)
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Equipartition of energy (1D) – Boltzmann (1896)

\[ \frac{d}{dt} \langle x^2 \rangle = \frac{2 k_B T}{\gamma} \left( 1 - \exp\left( - \frac{\gamma}{m} t \right) \right) \quad \Rightarrow \quad \frac{d}{dt} \langle x^2 \rangle \approx \frac{2 k_B T}{\gamma} \]

\[ t \gg \left( \frac{m}{\gamma} \right) \approx 10^{-8} \]
Langevin’s Equation (1908)

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\[
\Rightarrow \quad \frac{d}{dt} \left\langle x^2 \right\rangle \approx \frac{2k_B T}{\gamma}
\]

\[
\left\langle x^2 \right\rangle \sim 2Dt
\]

\[ t \gg \left( \frac{m}{\gamma} \right) \approx 10^{-8} \]
Random Walks

A man starts from a point $O$ and walks $\ell$ yards in a straight line; he then turns through any angle whatever and walks another $\ell$ yards in a second straight line. He repeats this process $n$ times. I require the probability that after these $n$ stretches he is at a distance between $r$ and $r + \delta r$ from his starting point, $O$.

Karl Pearson (1905)
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Karl Pearson (1905)

If $n$ be very great, the probability sought is

$$\frac{2}{n} e^{-r^2/n} r \, dr$$

Lord Rayleigh (1905)
The lesson of Lord Rayleigh’s solution is that in open country the most probable place to find a drunken man who is at all capable of keeping on his feet is somewhere near his starting point!

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... the consideration of true prices permits the statement of the fundamental principle – The mathematical expectation of the speculator is zero.

Louis Bachelier (1900)
Random Walks and the Binomial Distribution

A particle starts from an origin and at each time step $\Delta t$ the particle has an equal probability of jumping an equal distance $\Delta x$ to the left or the right. What is the probability $P_{m,n}$ that the particle will be at position $x = m\Delta x$ at time $t = n\Delta t$?
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$$P_{m,n} = \frac{1}{2} P_{m-1,n-1} + \frac{1}{2} P_{m+1,n-1},$$

$$P_{0,0} = 1, \quad \sum_{j=-k}^{k} P_{j,k} = 1 \quad k = 0, 1, 2, \ldots n$$
Enumeration

Suppose an $n$ step walk from 0 to $m$ has $k$ steps to the right and $n - k = k - m$ steps to the left.
Enumeration

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There are \( C(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \) ways of distributing these \( k \) steps among the \( n \) steps.
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\[ p(m(k), n) = \frac{C(n, k)}{2^n} \]
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There are $2^n$ possible paths in an $n$ step walk.

$p(m(k), n) = \frac{C(n, k)}{2^n}$

But $k = \frac{n + m}{2} \quad \Rightarrow \quad p(m, n) = \frac{n!}{2^n \left(\frac{n+m}{2}\right)! \left(\frac{n-m}{2}\right)!}$
Random Walks and the Normal Distribution

In $n$ step walks from 0 to $m$ the av. # steps to the right is
$$\langle k \rangle = \frac{n}{2}.$$  

How are fluctuations $X = k - \langle k \rangle = \frac{n + m}{2} - \frac{n}{2} = \frac{m}{2}$ distributed?
Random Walks and the Normal Distribution

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- $p(m(X), n) = \frac{n!}{2^n \left( \frac{n + m}{2} \right)! \left( \frac{n - m}{2} \right)!} = \frac{n!}{(\frac{n}{2} - X)! \left( \frac{n}{2} + X \right)! 2^n}$
Random Walks and the Normal Distribution

In \( n \) step walks from 0 to \( m \) the av. # steps to the right is \( \langle k \rangle = \frac{n}{2} \).

How are fluctuations \( X = k - \langle k \rangle = \frac{n + m}{2} - \frac{n}{2} = \frac{m}{2} \) distributed?

\[
p(m(X), n) = \frac{n!}{2^n (\frac{n+m}{2})! (\frac{n-m}{2})!} = \frac{n!}{(\frac{n}{2} - X)! (\frac{n}{2} + X)! 2^n}
\]

\( n! \approx \sqrt{2\pi n} n^n e^{-n} \) \( \Rightarrow \)

\[
P(X, n) = \frac{\sqrt{\frac{2}{n\pi}}}{(1 - \frac{2X}{n})^{(\frac{n}{2} - X + \frac{1}{2})} (1 + \frac{2X}{n})^{(\frac{n}{2} + X + \frac{1}{2})}}
\]
Re-write

\[ P(X, n) = \sqrt{\frac{2}{n\pi}} \exp \left[ \left( \frac{n}{2} - X + \frac{1}{2} \right) \ln \left( 1 - \frac{2X}{n} \right) + \left( \frac{n}{2} + X + \frac{1}{2} \right) \ln \left( 1 + \frac{2X}{n} \right) \right] \]

Carry out a series expansion of the log terms in powers of \( \frac{2X}{n} \)

\[ P(X, n) \sim \sqrt{\frac{2}{n\pi}} e^{\frac{-2X^2}{n}} = \sqrt{\frac{2}{n\pi}} e^{\frac{-m^2}{2n}} \]

Recall \( m \) is the final position of the walker.

The probability density function for unbiased standard random walks in the long time limit is the Gaussian or normal distribution.
Random Walks in the Continuum Approximation

Write \( P(m, n) = P(x, t), \quad x = m\Delta x, \quad t = n\Delta t \)

\[
P(x, t) = \frac{1}{2} P(x - \Delta x, t - \Delta t) + \frac{1}{2} P(x + \Delta x, t - \Delta t)
\]
Random Walks in the Continuum Approximation

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P(x, t) = \frac{1}{2} P(x - \Delta x, t - \Delta t) + \frac{1}{2} P(x + \Delta x, t - \Delta t)
\]

2. Taylor expansions

\[
P(x \pm \Delta x, t - \Delta t) \approx P(x, t) \pm \Delta x \frac{\partial P}{\partial x} - \Delta t \frac{\partial P}{\partial t} + \frac{1}{2} (\Delta x)^2 \frac{\partial^2 P}{\partial x^2} + \frac{1}{2} (\Delta t)^2 \frac{\partial^2 P}{\partial t^2}
\]

\[
\mp \Delta t \Delta x \frac{\partial^2 P}{\partial x \partial t} + O((\Delta t)^3) + O((\Delta x)^3),
\]
Random Walks in the Continuum Approximation

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Taylor expansions

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P(x \pm \Delta x, t - \Delta t) \approx P(x, t) \pm \Delta x \frac{\partial P}{\partial x} - \Delta t \frac{\partial P}{\partial t} + \frac{(\Delta x)^2}{2} \frac{\partial^2 P}{\partial x^2} + \frac{(\Delta t)^2}{2} \frac{\partial^2 P}{\partial t^2} + \Delta t \Delta x \frac{\partial^2 P}{\partial x \partial t} + O((\Delta t)^3) + O((\Delta x)^3),
\]

Retain leading order terms in \( \Delta t \) and \( \Delta x \) then

\[
\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2}, \quad D = \lim_{\Delta t \to 0, \Delta x \to 0} \frac{(\Delta x)^2}{2\Delta t} = \text{constant}
\]
Fundamental Solution and Mean Square Displacement

Green’s solution \( G(x, t), \quad G(x, 0) = \delta(x). \)
Fundamental Solution and Mean Square Displacement

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\[ \text{Fourier transform} \quad \frac{d\hat{G}(q, t)}{dt} = -Dq^2 \hat{G}(q, t), \quad \hat{G}(q, 0) = \hat{\delta}(q) = 1 \]
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  \]

- Fourier solution
  \[
  \hat{G}(q, t) = e^{-Dq^2 t}
  \]
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- Fourier solution
  \[ \hat{G}(q, t) = e^{-Dq^2t} \]

- Inverse Fourier transform
  \[ G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\frac{-Dq^2t + iqx}{dq}} dq \]
  \[ = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \]
Fundamental Solution and Mean Square Displacement

Green’s solution \( G(x, t), \quad G(x, 0) = \delta(x) \).

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- Inverse Fourier transform

\[
G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\underbrace{-Dq^2t + iqx}_{\text{complete the square}}} dq = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}
\]

- Mean square displacement

\[
\langle x^2 \rangle = \int_{-\infty}^{+\infty} x^2 G(x, t) \, dx \quad \text{or} \quad \langle x^2 \rangle = \lim_{q \to 0} -\frac{d^2}{dq^2} \hat{G}(q, t) = 2Dt.
\]
Central Limit Theorem

Each step of a random walk is a random variable $\Delta x_i$. The mean of $\Delta x_i$ after $N$ steps is the position $x$ divided by $N$. 
Central Limit Theorem

- Each step of a random walk is a random variable \( \Delta x_i \). The mean of \( \Delta x_i \) after \( N \) steps is the position \( x \) divided by \( N \).

- If we sample (infinitely many) random walks of length \( N \) (sufficiently large) then the sampling distribution of random variables \( X = \frac{x}{N} \) should approach a normal distribution

\[
P(X \in dx) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\]

with \( \mu = \langle X \rangle \) equal to the mean of the sampled set and variance \( \sigma^2 = \langle X^2 \rangle - \langle X \rangle^2 \) equal to the variance of the sampled set divided by \( N \).
Central Limit Theorem

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with $\mu = \langle X \rangle$ equal to the mean of the sampled set and variance $\sigma^2 = \langle X^2 \rangle - \langle X \rangle^2$ equal to the variance of the sampled set divided by $N$.

- Consider $\mu = 0$ and $\sigma^2 = 2Dt$. 

Diffusion Equation

If there are \( N \) non-interacting walkers then they all have the same probability \( P(x, t) \) of being at \( x \) at time \( t \) and hence the number per unit volume (concentration) at \( x \) at time \( t \) is \( c(x, t) = NP(x, t) \)

\[
\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}
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Diffusion Equation

If there are $N$ non-interacting walkers then they all have the same probability $P(x, t)$ of being at $x$ at time $t$ and hence the number per unit volume (concentration) at $x$ at time $t$ is $c(x, t) = NP(x, t)$

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}$$

The diffusion equation can be derived from conservation of mass and Fick’s Law.

$$\frac{\partial c}{\partial t} = -\frac{\partial q}{\partial x}$$

conservation of mass

$q = -\frac{\partial c}{\partial x}$

Fick’s law
Fick’s Law (1855)

Define $q(x, t)$ as the number of particles per unit time that pass through a test area perpendicular to the flow in the positive $x$ direction.
Fick’s Law (1855)

Define \( q(x, t) \) as the number of particles per unit time that pass through a test area perpendicular to the flow in the positive \( x \) direction.

The net flow of diffusing particles is from regions of high concentration to regions of low concentration and the magnitude of this flow is proportional to the concentration gradient.

\[
q(x, t) = -D \frac{\partial c}{\partial x}
\]
Conservation of Mass

\[
\begin{pmatrix}
\text{number of particles} \\
\text{in volume } V \text{ at time } t + \delta t
\end{pmatrix}
= 
\begin{pmatrix}
\text{number of particles} \\
\text{in volume } V \text{ at time } t
\end{pmatrix}
+ 
\begin{pmatrix}
\text{net number of particles} \\
\text{entering volume } V \text{ between times } t, t + \delta t
\end{pmatrix}
\]

\[
(c(x, t + \delta t)A\delta x) = (c(x, t)A\delta x) + (q(x, t)A\delta t - q(x + \delta x, t)A\delta t)
\]

\[
\Rightarrow \frac{c(x, t + \delta t) - c(x, t)}{\delta t} = -\frac{q(x + \delta x) - q(x, t)}{\delta x}
\]

\[
\Rightarrow \frac{\partial c}{\partial t} = -\frac{\partial q}{\partial x}
\]
Generalizations
If the media is spatially heterogeneous then replace the diffusion constant in Fick’s law with a space dependent function.

\[
\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left( D(x) \frac{\partial c}{\partial x} \right)
\]
diffusion 1D
Generalizations

If the media is spatially heterogeneous then replace the diffusion constant in Fick’s law with a space dependent function.

\[
\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left( D(x) \frac{\partial c}{\partial x} \right) \quad \text{diffusion 1D}
\]

\[
\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left( D(x) \frac{\partial c}{\partial x} \right) - \frac{\partial}{\partial x} (v(x) c) \quad \text{advection-diffusion 1 dim}
\]
**Generalizations**

If the media is spatially heterogeneous then replace the diffusion constant in Fick’s law with a space dependent function.

\[
\begin{align*}
\frac{\partial c}{\partial t} &= \frac{\partial}{\partial x} \left( D(x) \frac{\partial c}{\partial x} \right) \quad \text{diffusion 1D} \\
\frac{\partial c}{\partial t} &= \frac{\partial}{\partial x} \left( D(x) \frac{\partial c}{\partial x} \right) - \frac{\partial}{\partial x} (v(x) c) \quad \text{advection-diffusion 1 dim} \\
\frac{\partial c}{\partial t} &= \frac{\partial}{\partial x} \left( D(x) \frac{\partial c}{\partial x} \right) - \chi \frac{\partial}{\partial x} \left( c \frac{\partial u}{\partial x} \right) \quad \text{cematactic-diffusion 1 dim}
\end{align*}
\]
Generalizations

If the media is spatially heterogeneous then replace the diffusion constant in Fick’s law with a space dependent function.

\[
\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left( D(x) \frac{\partial c}{\partial x} \right) \quad \text{diffusion 1D}
\]

\[
\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left( D(x) \frac{\partial c}{\partial x} \right) - \frac{\partial}{\partial x} \left( v(x) c \right) \quad \text{advection-diffusion 1 dim}
\]

\[
\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left( D(x) \frac{\partial c}{\partial x} \right) - \chi \frac{\partial}{\partial x} \left( c \frac{\partial u}{\partial x} \right) \quad \text{chemotactic-diffusion 1 dim}
\]

\[
\frac{\partial c}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} D(r) \frac{\partial c}{\partial r} \right) \quad \text{radially symmetric d dim}
\]
Master Equations and the Fokker-Planck Equation

\[ P(x, t) = \int_{-\infty}^{+\infty} \lambda(\Delta x) P(x - \Delta x, t - \Delta t) d\Delta x \]

- \( P(x, t) \) probability density for walker to be at \( x \) at time \( t \)
- \( \lambda(\Delta x) \) probability density function for a jump of length \( \Delta x \)
- \( \Delta t \) discrete time between jumps

For spatially heterogeneous media generalize \( \lambda = \lambda(\Delta x, x) \) the step length density varies with position

\[ P(x, t) = \int_{-\infty}^{+\infty} \lambda(\Delta x, x - \Delta x) P(x - \Delta x, t - \Delta t) d\Delta x \]
Continuum limit $\Delta t \to 0$ and $\Delta x \to 0$

$$P|_{(x, t-\Delta t)} + \Delta t \frac{\partial P}{\partial t} \bigg|_{(x, t-\Delta t)} \approx \int_{-\infty}^{+\infty} \left( \lambda|_{(\Delta x, x)} - \Delta x \frac{\partial \lambda}{\partial x} \bigg|_{(\Delta x, x)} + \frac{\Delta x^2}{2} \frac{\partial^2 \lambda}{\partial x^2} \bigg|_{(\Delta x, x)} \right) \times \left( P|_{(x, t-\Delta t)} - \Delta x \frac{\partial P}{\partial x} \bigg|_{(x, t-\Delta t)} + \frac{(\Delta x)^2}{2} \frac{\partial^2 P}{\partial x^2} \bigg|_{(x, t-\Delta t)} \right)$$

$$\int_{-\infty}^{+\infty} \lambda(\Delta x, x) \, d\Delta x = 1,$$

$$\int_{-\infty}^{+\infty} \Delta x \lambda(\Delta x, x) \, d\Delta x = \langle \Delta x(x) \rangle,$$

$$\int_{-\infty}^{+\infty} \Delta x^2 \lambda(\Delta x, x) \, d\Delta x = \langle \Delta x^2(x) \rangle.$$
General Fokker-Planck Equation

\[ \frac{\partial P}{\partial t} = \frac{\partial^2}{\partial x^2} \left( D(x) P(x, t) \right) - \frac{\partial}{\partial x} \left( v(x) P(x, t) \right) \]

\[ v(x) = \frac{\langle \Delta x(x) \rangle}{\Delta t} \text{drift} \]

\[ D(x) = \frac{\langle \Delta x^2(x) \rangle}{2\Delta t} \text{diffusivity} \]
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\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial x^2} \left( D(x) P(x, t) \right) - \frac{\partial}{\partial x} \left( v(x) P(x, t) \right)
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\[v(x) = \frac{\langle \Delta x(x) \rangle}{\Delta t}\] drift

\[D(x) = \frac{\langle \Delta x^2(x) \rangle}{2\Delta t}\] diffusivity

Compare the generalized Fickian diffusion equation

\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial x} \left( D(x) \frac{\partial P(x, t)}{\partial x} \right) - \frac{\partial}{\partial x} \left( v(x) P(x, t) \right)
\]
The Chapman-Kolmogorov Equation, Markov Processes

In the limit where the time increment approaches zero the sequence of jumps \( \{ X_t \} \) in a random walk defines a stochastic process. A realization defines a trajectory \( x(t) \).
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If Markov then the conditional probability \( q(x, t|x', t') \) that \( X_t \) lies in \( x, x + dx \) given that \( X_{t'} \) lies in \( x', x' + dx' \) satisfies

\[
q(x, t|x'', t'') = \int q(x, t|x', t')q(x', t'|x'', t'') \, dx'
\]

Bachelier (1990)

Note the times \( t > t' > t'' \) are discrete.
Wiener Process, Brownian Motion

One solution of Bachelier’s equation is

\[ q(x, t|x', t') = \frac{1}{\sqrt{2\pi(t-t')}} e^{-\frac{(x-x')^2}{2(t-t')}} , \quad t > t' \]

Wiener process
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- Limiting behaviour of a random walk in the limit where the time increment approaches zero (Wiener, 1923) – also referred to as Brownian motion $B_t$. 
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- Limiting behaviour of a random walk in the limit where the time increment approaches zero (Wiener, 1923) – also referred to as Brownian motion \( B_t \).

- (i) Realizations \( x_B(t) \) of \( B_t \) are continuous but nowhere differentiable – \( x_B(t) \) versus \( t \) is a fractal graph with dimension \( d = 3/2 \). (ii) The increments \( B_t - B_{t'} \) are normally distributed with mean 0 and variance \( t - t' \) for \( t > t' \). (iii) The increments \( B_t - B_{t'} \) and \( B_s - B_{s'} \) are independent for \( t > t' \geq s \geq s' \geq 0 \).
Standard diffusion

- The mean square displacement scales linearly with time and the probability density function is the Gaussian normal distribution
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- Theoretical support from random walks, central limit theorem, Langevin equation, diffusion equation, master equations, Weiner processes.
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- Theoretical support from random walks, central limit theorem, Langevin equation, diffusion equation, master equations, Weiner processes.

- Experimental support widespread, e.g., Perrin (1909) measured mean square displacements and used Einstein’s relations to determine Avogadro’s number, the constant number of molecules in any mole of substance, thus consolidating the atomistic description of nature.
In the theory of Brownian motion the first concern has always been the calculation of the mean square displacement of the particle, because this could be immediately observed.

George Uhlenbeck and Leonard Ornstein (1930)
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Geoge Uhlenbeck and Leonard Ornstein (1930)

In the atmosphere a spreading dot will not serve as an element from which general distributions can be built up. 
Lewis Richardson (1926)
Anomalous Diffusion

There have been numerous experimental measurements of anomalous diffusion – the mean square displacement does not scale linearly with time.

\[ \langle \Delta X^2(t) \rangle = \left\langle (X(t) - \langle X(t) \rangle)^2 \right\rangle \sim t \]
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Anomalous diffusion is ‘normal’ in spatially disordered systems, porous media, fractal media, turbulent fluids and plasmas, biological media with traps, binding sites or macro-molecular crowding, stock price movements.
## Anomalous Scaling

<table>
<thead>
<tr>
<th>$\langle \Delta X^2 \rangle \sim t (\ln t)^\kappa$</th>
<th>$1 &lt; \kappa &lt; 4$</th>
<th>ultraslow diffusion</th>
<th>Sinai diffusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \Delta X^2 \rangle \sim t^\alpha$</td>
<td>$0 &lt; \alpha &lt; 1$</td>
<td>subdiffusion</td>
<td>disordered solids</td>
</tr>
<tr>
<td>$\langle \Delta X^2 \rangle \sim \begin{cases} t^\alpha &amp; t &lt; \tau \ t &amp; t &gt; \tau \end{cases}$</td>
<td>transient subdiffusion</td>
<td>biological media</td>
<td></td>
</tr>
<tr>
<td>$\langle \Delta X^2 \rangle \sim t$</td>
<td>standard diffusion</td>
<td>homogeneous media</td>
<td></td>
</tr>
<tr>
<td>$\langle \Delta X^2 \rangle \sim t^\beta$</td>
<td>$1 &lt; \beta &lt; 2$</td>
<td>superdiffusion</td>
<td>turbulent plasmas</td>
</tr>
<tr>
<td>$\langle \Delta l^2 \rangle \sim t^3$</td>
<td>Richardson diffusion</td>
<td>atmospheric turbulence</td>
<td></td>
</tr>
</tbody>
</table>
Fractional Diffusion

Over the past decade a new theoretical framework has been developed to model anomalous diffusion. The new framework is based around the physics of continuous time random walks and the mathematics of fractional calculus.

One can ask what would be a differential having as its exponent a fraction. Although this seems removed from Geometry . . . it appears that one day these paradoxes will yield useful consequences.

Gottfried Leibniz (1695)