Probing structure and dynamics of exotic nuclei using fast one- and two-nucleon knockout reactions

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Abstract

Sudden two nucleon removal reactions on light, nuclear targets offer an excellent probe of the properties of exotic nuclei and are ideally suited to studying the evolution of nuclear structure far from stability. Here we extend the theoretical formalism for two-nucleon knockout to calculate heavy residue longitudinal momentum distributions, which are shown to be a sensitive probe of the total angular momentum of the nucleon pair. In tandem with $\gamma$-ray spectroscopy, residue momentum distribution measurements provide a powerful tool for the identification of final state spins in very exotic systems. The sensitivity of the residue momentum distributions with respect to the nucleon binding energy and other factors is investigated.

Two-nucleon angular correlations are discussed, indicating that the total orbital angular momentum is key to determining the residue momentum distribution shape. In addition, this provides a further insight regarding subtle differences in the residue momentum distribution associated with the particular quantum numbers of the individual nucleons.

Experimental sd-shell examples ($^{22}$Mg(−2n), $^{38}$Si(−2p) and $^{28}$Mg(−2p)) are confronted and excellent agreement is obtained with experimentally measured residue momentum distributions. The application of knockout reactions to heavy systems is considered, using two-proton removal from $^{208}$Pb as an example. Experimental isomeric ratios for this reaction are reasonably reproduced and the complications in their calculation, due to the high density of residue states and absence of prompt $\gamma$-decay measurements, are highlighted. We also emphasise the importance of complete transmission of the full longitudinal momentum distribution of the residues in experiments measuring the isomeric ratios of high-spin states.
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“Peculiar travel suggestions are dancing lessons from God.”
Cat’s Cradle, Kurt Vonnegut
Declaration of Originality

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Edward Simpson
3.4 Details of computer codes .................................. 47

4 Angular correlations ........................................ 49
  4.1 Angular correlation function in $jj$-coupling ............. 50
  4.2 Angular correlations in $LS$ coupling ...................... 55
  4.3 Momentum distributions with angular separation ........... 59

5 Approximations for $R_{j\lambda}^i$ ................................. 62
  5.1 Properties of $R_{j\lambda}^i$ .................................. 62
  5.2 Large $\ell$ approximations for $Y_{\ell\lambda}(\theta,\varphi)$ ......... 63
  5.3 Gaussian wave function .................................. 65
    5.3.1 Explicit case of $\ell = 2$ .............................. 67
    5.3.2 General case for $\lambda = \ell$ .......................... 68
    5.3.3 Full residue distribution for the maximally aligned case .. 68
  5.4 Hankel wave function .................................. 69
  5.5 Comparisons of approximations for $R_{2\lambda}^j$ ............. 71

6 Sensitivities .................................................. 74
  6.1 Nucleon pair total angular momentum ....................... 75
  6.2 Final state alignment .................................. 77
  6.3 Nucleon binding energy .................................. 81
  6.4 Diffractive-stripping contributions ........................ 81
  6.5 Centre of mass impact parameter .......................... 86
  6.6 Total orbital angular momentum ........................... 89

7 Applications .................................................. 95
  7.1 Examples from the $sd$–shell .............................. 95
    7.1.1 Two-neutron knockout from $^{22}$Mg .................... 96
    7.1.2 Two-proton knockout from $^{38}$Si ...................... 101
    7.1.3 Two-proton knockout from $^{28}$Mg ...................... 104
  7.2 Heavy projectiles ....................................... 107
    7.2.1 Calculation details .................................. 110
    7.2.2 Isomeric ratio as a function of residue momentum ....... 116
    7.2.3 Example momentum distributions ....................... 121

8 Conclusions .................................................. 126
  8.1 Summary ............................................... 126
  8.2 Topics of further interest ............................... 127
    8.2.1 Heavy systems ................................... 127
CONTENTS

8.2.2 Isomeric beams ................................................. 128
8.2.3 Unlike pair knockout ......................................... 128
8.2.4 Deformed systems ............................................. 129
8.2.5 Weakly populated excited states ............................. 130

A Two-step reactions with thick targets ........................ 132

B Peer-reviewed publications ..................................... 134
## List of Figures

2.1 Composite projectile co-ordinates .................................. 7  
2.2 Angular momentum coupling scheme .................................. 16  
2.3 Co-ordinate system ....................................................... 16  
2.4 Surface localisation ..................................................... 22  
2.5 Momentum components .................................................. 32  
4.1 Angular separation function for \([0d_{5/2}]^2\) .................................. 53  
4.2 Angular separation function for \([\ell_{\ell+1/2}]^2, I = 0\) .................. 54  
4.3 Angular separation function for \([\ell_{\ell+1/2}]^2, I = 4\) .................. 55  
4.4 Partial angular correlation function in LS coupling .................. 58  
4.5 Nucleon angular separation ............................................ 60  
4.6 Angular separation momentum distributions .......................... 61  
5.1 \(R_{\ell\lambda}(s = 8, \kappa)\) for \(\ell = 5\) and even \(\lambda\) .................. 65  
5.2 \(R_{\ell\lambda}(s = 8, \kappa)\) for \(\ell = 5\) and odd \(\lambda\) .................. 66  
5.3 Approximate \(\ell = 2\) wave functions for \(^{22}\text{Mg}\) ................. 71  
5.4 Comparison of approximations for \(R_{2\lambda}(s, \kappa)\) .................. 72  
6.1 Pair coupling differences .............................................. 76  
6.2 Uncorrelated knockout .................................................. 77  
6.3 Final state projection ................................................... 79  
6.4 Residue momentum distribution for maximum alignment .......... 80  
6.5 Sensitivity to nucleon binding energy ................................ 82  
6.6 Diffraction cross section .............................................. 84  
6.7 Localisation of diffraction ............................................ 85  
6.8 Diffractive-stripping residue momentum distributions ............. 87  
6.9 Sensitivity to minimum centre of mass impact parameter .......... 88  
6.10 Sensitivity to total orbital angular momentum ..................... 90  
6.11 \([0d_{5/2}]^2\) and \([0d_{5/2}][0d_{3/2}] 4^+\) states in LS coupling ........... 92  
6.12 \([0d_{5/2}]^2\) and \([0d_{5/2}]^2 2^+\) states in LS coupling ................. 93
7.1 $^{22}\text{Mg}$ ($-2n$) levels schematic ........................................... 97
7.2 $^{22}\text{Mg}$ ($-2n$) momentum distributions ............................... 100
7.3 $^{38}\text{Si}$ ($-2p$) levels schematic ........................................ 102
7.4 $^{38}\text{Si}$ ($-2p$) momentum distributions ............................... 104
7.5 $^{28}\text{Mg}$ ($-2p$) levels schematic ........................................ 105
7.6 $^{28}\text{Mg}$ ($-2n$) momentum distributions ............................... 108
7.7 $^{206}\text{Hg}$ decay scheme ................................................... 111
7.8 $^{208}\text{Pb}$ separation energies schematic .................................. 112
7.9 $^{208}\text{Pb}$($-2p$) momentum distributions ............................... 118
7.10 Isomeric ratios as a function of residue momentum cut ............... 119
7.11 Experimentally broadened $^{206}\text{Hg}$ momentum distribution ......... 121
7.12 $^{208}\text{Pb}$($-2p$) momentum distributions for $[0h_{11/2}]^2$ and $[2s_{1/2}]^2$ .... 123
7.13 $^{208}\text{Pb}$($-2p$) alignment for $[0h_{11/2}]^2$, $I = 10$ ......................... 124
7.14 $^{208}\text{Pb}$($-2p$) momentum distributions for $3^+$ states ............... 125
Chapter 1

Introduction

1.1 Motivation

The development of radioactive beams has revolutionised the field of nuclear physics, providing beams of highly exotic radioactive nuclei. For the first time, highly unstable nuclei, far from the valley of $\beta$-stability have become available for reaction studies. Topics of particular interest are the limits of nuclear stability (drip-lines, nuclear halos), the evolution of shell structure far from stability (e.g. the islands of inversion) and the (associated) quenching of magic numbers. Exotic nuclei offer new and strenuous tests of nuclear structure models and are particularly important to nuclear astrophysics, as nuclei far from stability determine the paths of the astrophysical processes that create the elements.

One method of producing radioactive ion beams is in-flight fragmentation. A primary driver beam is focussed onto a production target producing a vast array of reaction products. One particular reaction product, the desired secondary beam, is then selected using a series of magnets and is focussed onto a second reaction target, at which the reaction of interest occurs. Following the reaction target further sets of magnets identify and select the final reaction residues, the momentum of which can be measured. The relatively low-intensity of the secondary beam allows one to surround the reaction target with $\gamma$-ray detectors to perform in-beam spectroscopy on the desired reaction residues. The driver beam (and therefore secondary beam) energy largely depends on the experimental facility, but the class of reactions that will be of interest here are best described at 60 MeV/nucleon or greater, with the theoretical approximations made improving as the energy increases. Within the next decade a number of new radioactive beam facilities will be commissioned; the Radioactive Ion Beam Factory (RIBF) at RIKEN is already in operation, and the Facility for Rare Isotope Beams (FRIB) at Michigan State University and the Super Fragment Separator at Helmholtzzentrum für Schwerionenforschung GmbH (GSI) are under construction. These facilities will provide intense radioactive beams cov
ering vast swathes of the nuclear chart, currently *terra incognita*, at energies of $150 - 1000$ MeV/nucleon.

A particular class of reactions that has been extensively exploited at current radioactive beam facilities involves the sudden removal of a single nucleon by a light nuclear target [1], leaving behind an exotic heavy residue. Such reactions provide an efficient method for producing exotic nuclei, which can be studied through $\gamma$-spectroscopy and via measurement of the momentum of the reaction residue. The strength of the technique lies in the experimental efficiency in measuring the residues. The nucleon is either removed via an elastic (diffraction) or inelastic (absorption) interaction with the target, but is typically not observed in experiments. The final state exclusive cross sections of such reactions allow the extraction of single nucleon spectroscopic strengths [2] and the investigation of the suppression of such spectroscopic strength in asymmetric systems [3, 4]. In addition, the momentum distribution of the heavy residue is characteristic of the orbital angular momentum of the removed nucleon [1, 5, 6]. These facets of the reaction mechanism have been used to investigate the ground state properties and low lying structure of exotic nuclei with mass $A < 60$.

More recently the sudden removal of two well-bound nucleons has been exploited to produce very specific exotic $(N,Z)$ combinations. The first description of the removal of two well-bound nucleons as a direct reaction was given by Bazin *et al.* [7], with a more complete description provided by Tostevin *et al.* [8, 9]. Whereas single-nucleon removal tracks changes in single-particle structure, two-nucleon removal probes two-nucleon correlations and facilitates study of rapid changes in structure. Of particular recent interest are nuclei in and around the $^{31}$Na-centred *island of inversion* [10], where two-particle-two-hole neutron configurations dominate the ground state wave functions due to a tensor force reduced neutron $sd - pf$ shell gap. Structural changes in $^{32}$Mg$\rightarrow^{30}$Ne [11] and $^{38}$Si$\rightarrow^{36}$Mg [12] have been investigated via two-proton knockout, and the non-magic nature of $^{42}$Si has been clarified in similar studies [13, 14]. Further experiments have investigated a new *island of inversion* associated with $N \approx 40$ [15]. Such experiments deduce rapid changes in structure from highly suppressed two-nucleon removal cross sections. In a similar manner to the spectroscopic factors deduced in single-nucleon removal experiments, we require an accurate theoretical description of the final-state exclusive cross sections, as provided by Ref. [9]. This eikonal, adiabatic model incorporates both inelastic (stripping) and elastic (diffraction) nucleon removal reaction mechanisms and reproduces the experimental final-state exclusive cross sections for $^{28}$Mg$(-2p)$, $^{54}$Ti$(-2p)$, $^{26}$Si$(-2p)$, $^{30}$S$(-2p)$ and $^{34}$Ar$(-2n)$.

The great value of the single-nucleon removal technique is not only that precise spectroscopic information can be obtained from absolute cross section, but that the shape
of the residue longitudinal (beam direction) momentum distribution is characteristic of
the orbital angular momentum of the removed nucleon [5]. The measurement of such
momentum distributions is commonplace and is a valuable facet of the technique. This
aspect of two-nucleon removal requires further theoretical development. Approximate
calculations based on limited sampling of two-nucleon overlap functions have indicated
that the residue momentum distribution may be strongly sensitive to the total angular
momentum of the two-nucleons [16, 17]. The primary objective of this thesis is the ex-
tension of current theoretical techniques to calculate the residue momentum distributions
following the sudden removal of two nucleons from a fast radioactive projectile, and to
provide an understanding of the extent to which these are sensitive to or could elucidate
the correlations of nucleon pairs.

1.2 Outline

As stated, the primary objective is the theoretical framework for, and the sensitivities
of, residue momentum distributions following the sudden removal of two nucleons. In
Chapter 2 we discuss briefly the underlying (eikonal, adiabatic) approximations, before
considering the two-nucleon removal cross section, which is then developed to consider
residue momentum distributions. Careful attention is paid to the projectile-surface local-
isation of the cross section and the consequences for events where one nucleon interacts
elastically with the target. We then pursue simplifications to the formalism developed,
in order to better understand its sensitivities. In Chapter 4 we investigate two-nucleon
angular correlations, gaining insight into the sensitivity of the residue momentum dis-
tribution to the particular quantum numbers of the nucleon pair. We then make some
approximations to the nucleon radial wave function, giving insight into the sensitivities of
the nucleon intrinsic momentum distribution. The sensitivity of the residue momentum
distribution to various factors is investigated in Chapter 6, with the primary sensitivity
being to the total angular momentum of the two nucleons. This makes final-state spin
assignments possible for cases where the projectile is even-even. Experimental exam-
pies are then confronted in Chapter 7, where excellent agreement is found for $^{22}\text{Mg}(-2n)$,
$^{38}\text{Si}(-2p)$ and $^{28}\text{Mg}(-2p)$. We proceed to consider two-proton removal from a heavy mass
projectile, using $^{208}\text{Pb}(-2p)$ as a test case, for which there are recently acquired data.
Isomeric ratios for states populated in $^{206}\text{Hg}$ were measured in a recent experiment [18],
and reasonable agreement is obtained. In addition we discuss the residue momentum dis-
tributions and the importance of understanding the experimental longitudinal momentum
transmission for determining isomeric ratios for high-spin isomers. We then summarise
the principle results and discuss possibilities for future work in the concluding Chapter.
1.2. OUTLINE

A significant proportion of the key results of this thesis are now published in two peer-reviewed articles, Refs. [19] and [20], which described the theoretical formalism itself and the associated sd-shell examples mentioned above. These articles can be found in Appendices B.2 and B.3 respectively. The example of $^{208}\text{Pb}(-2p)$ has been published in Physical Review C, and this article can be found in Appendix B.4. Further work on the additional complications that arise in the case of the removal of two weakly bound nucleons can be found in Appendix B.1. Specifically, the systematics of one- and two-neutron removal cross sections for neutron-rich carbon isotopes were studied. This article discusses the competing direct (single event two-nucleon removal) and indirect (single-nucleon removal to particle unbound states) mechanisms that must be considered when the removed nucleons are weakly bound, showing that the indirect mechanism was expected to dominate the two-nucleon removal cross section. This work was the first to quantify these issues, which are of significant importance to future studies of two-neutron correlations in the most neutron-rich systems.
Chapter 2

Two-nucleon knockout reactions

The objective here is the calculation of the heavy residue momentum distribution following the sudden removal of two like nucleons from a radioactive projectile by a light nuclear target. The theory of two-nucleon knockout reactions is developed within an eikonal, adiabatic framework and we begin by briefly summarising the approach and underlying approximations used in calculating the elastic scattering $S$-matrices. Static-density $S$-matrices will be used in all calculations, the primary input to which are the projectile and target point nucleon densities. The formalism for two-nucleon removal is described, closely following the Refs. [8] and [9]. This formalism is then extended for calculation of the residue longitudinal (beam directional) momentum distributions, paying careful attention to the contributions arising from events where one nucleon is removed in an elastic process and the other is absorbed by the target.

2.1 Elastic scattering $S$-matrices

2.1.1 Eikonal approximation for point particles

The eikonal approximation provides a semi-classical solution to the scattering problem, applicable at high energies where i) the depth of the scattering potential is much smaller than the center of mass energy, and ii) the wavelength of the particle is much shorter than the distance of significant variation of the potential. We introduce the eikonal approximation by considering the scattering of a point (neutral) particle by a central potential $V(R)$. Solutions to this problem under high-energy approximations have been discussed in the literature e.g. Refs. [21] and [22]. Our objective is to solve the time-independent Schrödinger equation,

$$\left[-\frac{\hbar^2}{2\mu} \nabla_R^2 + V(R) - E_{cm}\right] \psi_R(\vec{R}) = 0. \quad (2.1)$$
Here \( \vec{R} \) denotes the separation of the two particles, written in cylindrical coordinates as \( \vec{R} = \vec{b} + \hat{K} Z \), such that the direction of the \( Z \)-axis \( \hat{Z} \) is aligned with beam direction \( \hat{K} \). The reduced mass is \( \mu \) and the centre of mass energy is \( E_{\text{cm}} \). We extract the incident plane wave from the scattering wave function and hence write it as a product of an incident plane wave and an as yet unknown modulation function \( \omega(\vec{R}) \),

\[
\psi_{\vec{K}}(\vec{R}) = \exp(i\vec{K} \cdot \vec{R}) \omega(\vec{R}).
\]  

(2.2)

Here \( K = (2\mu E_{\text{cm}}/\hbar^2)^{1/2} \) is the wavenumber. The classical velocity \( \nu \) is related to the wavenumber by \( \nu = \hbar K/\mu \). The modulation function \( \omega(\vec{R}) \) contains all effects of the projectile-target potential and is assumed to vary slowly on the scale of the projectile wavelength. Substituting this solution back into the Schrödinger equation and rearranging gives

\[
\left[ 2i\vec{K} \cdot \nabla_R \omega(\vec{R}) - \frac{2\mu}{\hbar^2} V(R)\omega(\vec{R}) + \nabla_R^2 \omega(\vec{R}) \right] \exp(i\vec{K} \cdot \vec{R}) = 0.
\]  

(2.3)

We make the approximations [21] that i) the strength of the potential is weak compared to the centre of mass energy, \( |V|/E_{\text{cm}} \ll 1 \), and ii) the wavelength of the particle is short compared to the distance \( a \) over which the potential appreciably changes, such that \( Ka \gg 1 \). We assume that \( \omega \) is expected to vary little over the particle wavelength. The eikonal approximation then neglects the second derivative (curvature) term \( \nabla_R^2 \omega(\vec{R}) \), which for large \( \vec{K} \) and a slowly varying potential, is smaller than \( 2\vec{K} \cdot \nabla_R \omega(\vec{R}) \) and thus we obtain the equation for the modulation function

\[
\frac{\partial \omega}{\partial Z} = -\frac{i}{\hbar \nu} V(R)\omega(\vec{R}).
\]  

(2.4)

This can be solved, assuming the incident plane wave boundary condition, that \( \omega(\vec{R}) \rightarrow 1 \) as \( \vec{R} \rightarrow -\hat{K} \infty \), to find the modulation function

\[
\omega(\vec{R}) = \exp \left[ -\frac{i}{\hbar \nu} \int_{-\infty}^{Z} dZ' V(R') \right].
\]  

(2.5)

The modulation function \( \omega(\vec{R}) \) is simply a one dimensional integral of the potential along the beam direction \( \hat{Z} \), that acquires a phase due to the potential field. Our neglect of the curvature term \( \nabla_R^2 \omega(\vec{R}) \) implies that the modulation function \( \omega(\vec{R}) \) can be accurately approximated by assuming that the particle follows a straight line path through the potential at fixed impact parameter \( b \). After the reaction where \( Z \rightarrow \infty \), we obtain the
2.1 ELASTIC SCATTERING S-MATRICES

Figure 2.1: Co-ordinates for the composite two-body \((c, \nu)\) projectile and target system. The projectile momentum is denoted by \(\vec{K}\). The internal projectile co-ordinate \(\vec{r}_i\) is fixed for the duration of the interaction with the target due to the adiabatic approximation.

asymptotic form for the eikonal approximation to the wave function,

\[
\lim_{Z \to \infty} \psi_K(\vec{R}) = S(b) \exp(i\vec{K} \cdot \vec{R}).
\] (2.6)

Here we have defined the eikonal approximation to the \(S\)-matrix \(S(b)\)

\[
S(b) = \exp \left[ -\frac{i}{\hbar \nu} \int_{-\infty}^{+\infty} dZ' V(R') \right],
\] (2.7)

which is essentially the amplitude for forward going scattered waves.

2.1.2 Two-body projectile

Here, the interest will be composite projectiles, where the projectile is described as several distinct constituents. The constituents can be individual nucleons or clusters of nucleons. We will consider a two-body projectile \(p\) consisting of a core \(c\) and valence nucleon \(\nu\). Here, \(\vec{R} = \vec{b} + Z \vec{K}\) denotes the separation of the target and projectile centre of mass, and \(\vec{r}\) denotes the separation of the projectile constituents. The vector connecting the projectile centre of mass and the projectile constituent \((c\ or \ \nu)\) is denoted by \(\vec{r}_i\). The co-ordinates for the composite projectile are shown in Fig. 2.1. The impact parameters of each projectile constituent, that is, the projection of \(\vec{R} + \vec{r}_i\) on the impact parameter \((x, y)\) plane, are denoted by \(b_i\).

To generalise the point-particle case, the first step is the adiabatic approximation. This assumes that one degree of freedom changes more slowly than another, and as such, the slowly changing degree of freedom can be frozen for the duration of a fast interaction concerning the other degree of freedom. In the current case we assume that the internal
projectile co-ordinate $\vec{r}$ varies slowly compared to the projectile-target co-ordinate $\vec{R}$. That is, the internal motions of the projectile constituents are assumed slow compared with the projectile-target motion. The result is that the internal projectile co-ordinates are considered to be fixed for the duration of the interaction with the target.

We must now solve the adiabatic Schrödinger equation. Since the projectile energy is assumed large when compared to the internal energy of the projectile corresponding to the internal Hamiltonian $H_p$, it is reasonable to replace the internal energies that enter the collision dynamics (determined by $H_p$) with the (small) ground state binding energy $\epsilon_0$. Making the adiabatic approximation to $H_p$ defines the adiabatic approximation to the scattering wave function as a solution of the adiabatic Schrödinger equation,

$$
\left[T_R + U_{pt}(\vec{R}, \vec{r}) - (E_{cm} + \epsilon_0)\right] \Psi_{Ad}^{Ad}(\vec{R}, \vec{r}) = 0,
$$

(2.8)

where $T_R$ is the centre of mass kinetic energy. The interaction between the projectile and target $U_{pt}(\vec{R}, \vec{r})$ is the sum of interactions of the individual constituents in the projectile and the target $V_{it}(|\vec{R} + \vec{r}_i|)$.

We then use an eikonal factorization of the adiabatic wave function, which separates the centre of mass motion and internal separation co-ordinates of the two-body projectile. The projectile internal (ground state) wave function is $\phi_0(\vec{r})$ and again the modulation function is $\omega(\vec{R}, \vec{r})$,

$$
\Psi_{Ad}^{Ad}(\vec{R}, \vec{r}) = \exp(i\vec{K} \cdot \vec{R}) \phi_0(\vec{r}) \omega(\vec{R}, \vec{r})
$$

(2.9)

where now $\hbar K = \sqrt{2\mu(E_{cm} + \epsilon_0)}$.

The same procedure can be followed to obtain the solution for the two-body modulation function $\omega(\vec{R}, \vec{r})$, again with the incident plane-wave boundary condition that for fixed vector $\vec{r}$, $\omega(\vec{R}, \vec{r}) \to 1$ as $\vec{R} \to -\infty \hat{K}$,

$$
\omega(\vec{R}, \vec{r}) = \exp\left[-\frac{i}{\hbar \nu} \int_{-\infty}^{Z} dZ' U_{pt}(\vec{R}', \vec{r})\right].
$$

(2.10)

In the second line we have simply expressed the projectile-target interaction $U_{pt}(\vec{R}, \vec{r})$ as a sum of the interactions of the individual constituents with the target $V_{it}$. The modulation function $\omega(\vec{R}, \vec{r})$ thus factorises into the modulation functions for the individual constituents $c$ and $v$. Again taking the post-reaction limit $Z \to \infty \hat{K}$, we obtain the
eikonal solution to the adiabatic Schrödinger equation,

$$\lim_{Z \to \infty} \Psi_{K}(\vec{R}, \vec{r}) = S_{c}(b_{c}) S_{v}(b_{v}) \exp(i\vec{K} \cdot \vec{R}) \phi_{0}(\vec{r}),$$

(2.11)

where the core $c$ and valence $v$ impact parameters are $b_{c}$ and $b_{v}$. The result is that for each fixed $\vec{r}$ the $S$-matrix for the interaction of a two-body projectile with the target can be written as a product of the $S$-matrices of the individual projectile constituents at the appropriate impact parameters. The projectile $S$-matrix is obtained by integrating over the projectile ground state internal co-ordinate $r$. This fact has been widely exploited in the analysis of halo systems and, in particular, the extraction of root-mean-square radii from measured reaction cross sections, where taking proper account of the projectile’s structure as an extended object with a few separated constituents is of importance.

### 2.1.3 Density folding models

Calculation of elastic scattering $S$-matrices from a folding of projectile and target densities has been formulated from various viewpoints in the literature. When considering the collision of an individual particle with a nucleus, Glauber [21] assumes an independent particle model for the many-body nucleus, neglecting nucleon correlations. The square modulus of the ground state wave function may then be written as the product of single-particle densities. Czyż and Maximon [23] discuss scattering profiles, essentially the scattering probability for a projectile with impact parameter $b$, in terms of projectile and target densities. Several authors [24, 25] discuss how the single-scattering $t_{pp}$ approximation relates the nuclear optical potential to the projectile and target ground state densities. Common to all such approaches is the connection of tangible projectile and target density profiles and the appropriate elastic scattering $S$-matrices, via a folding with an effective nucleon-nucleon interaction.

The approach outlined here, following Refs. [26] and [27], has been used in the analysis of the reaction and nucleon-removal cross sections for exotic nuclei. The radii of loosely bound $^{11}$Be, $^{6}$He, $^{11}$Li and $^{14}$Be were obtained from reaction cross section measurements [26, 27] using the static density $S$-matrices discussed in the following. Ref. [28] studies the systematics of the reaction cross sections of carbon isotopes. The technique has been used extensively in the study of single-nucleon removal reactions on light nuclear targets, both for weakly bound and strongly bound nucleons. Precise absolute spectroscopic strengths for protons and neutrons in $^{12}$C and $^{16}$O have been extracted from single-nucleon removal cross sections [2]. Calculations for single-neutron knockout using static-density $S$-matrices have also been shown to be consistent with more microscopic nucleon-nucleon potential approaches [29]. More recently, $S$-matrices calculated using this method have accurately
reproduced the relative strengths of the elastic and inelastic breakup contributions to single-proton removal from $^9$C and $^8$B [30]. Heavy residue momentum distributions calculated with static-density $S$-matrices are in precise agreement with experiment (see e.g. $^8$B($-1p$) [6], $^{19}$C($-1n$) [31]). Recently similar approaches have been applied to deformed projectiles, where the orientation of the projectile symmetry axis is important [32, 33].

Here we consider the scattering of a projectile $p$ with target $t$, where the mass (proton, neutron) number of the projectile is denoted by $A_p (Z_p, N_p)$, and similarly for the target. The projectile and target are assumed to have point nucleon density profiles $\rho_p(r)$ and $\rho_t(r)$. The proton and neutron density profiles of an extended projectile will be, when required, referred to as $\rho^p_p(r)$ and $\rho^p_t(r)$. If the projectile is a single nucleon, the density is simply a delta function. As before, the projectile-target co-ordinate is denoted by $\vec{R} = (\vec{b}, Z)$, where the $Z$-axis is coincident with the beam direction such that $b$ is the projectile centre of mass impact parameter. The internal co-ordinates of the projectile and target are denoted by $\vec{r}_p$ and $\vec{r}_t$ respectively.

We will assume the projectile and target interact through complex effective interactions $V_{it}$, approximated from the projectile and target densities and an approximate effective nucleon-nucleon interaction $t_{NN}$ (in units of inverse distance) by the folding integral,

$$\frac{i}{\hbar \nu} V_{it}(\vec{R}) = \int d\vec{r}_i \int d\vec{r}_t \rho_i(r_i) \rho_t(r_t) t_{NN}(|\vec{R} + \vec{r}_i - \vec{r}_t|)$$

(2.12)

Having made the forward scattering approximation, the elastic $S$-matrix is given as (Ref. [27]), where $t_{NN}$ is assumed central and the densities $\rho(r)$ assumed spherical,

$$S(b) = \exp \left[ - \int_{-\infty}^{+\infty} dZ \int d\vec{r}_i t_{NN}(|\vec{R} + \vec{r}_i - \vec{r}_t|) \rho_p(r_p) \rho_t(r_t) \right].$$

(2.13)

For the case where the projectile is a nucleon, assumed to have point density $\rho_p(r_p) = \delta(r_p)$, the $S$-matrix is given as

$$S(b) = \exp \left[ - \int_{-\infty}^{+\infty} dZ \int d\vec{r}_i t_{NN}(|\vec{R} - \vec{r}_i|) \rho_t(r_i) \right].$$

(2.14)

The effective nucleon-nucleon interaction $t_{NN}$ is taken to be

$$t_{NN}(r) = \frac{\sigma_{NN}^{pl}}{2} (1 - i\alpha)g(r).$$

(2.15)

Here $\sigma_{NN}^{pl}$ is the isospin averaged free nucleon-nucleon total cross section appropriate for
2.1. ELASTIC SCATTERING S-MATRICES

the projectile target system, given by,

$$\sigma^pt_{NN} = \frac{(N_pN_t + Z_pZ_t)\sigma_{pp} + (N_pZ_t + N_pZ_t)\sigma_{pn}}{A_pA_t},$$  \hspace{1cm} (2.16)

which takes into account the difference between the $nn$ and $pn$ free nucleon-nucleon total cross sections. We assume that the neutron-neutron and proton-proton interactions are identical. The total cross sections themselves are taken from the parametrization given by Ref. [34]. Experimental free nucleon-nucleon cross sections were fitted to obtain,

$$\sigma_{np} = -70.67 - 18.18/\beta + 25.26/\beta^2 + 113.85\beta,$$ \hspace{1cm} (2.17)

$$\sigma_{pp} = \sigma_{nn} = 13.73 - 15.04/\beta + 8.76/\beta^2 + 68.67\beta,$$ \hspace{1cm} (2.18)

where the cross sections are in mb and $\beta = v/c$ is the projectile laboratory velocity in units of the speed of light. This is related to the projectile lab energy $E_{lab}$ in units of MeV/nucleon, by

$$\beta = \frac{v}{c} = \frac{\sqrt{E_{lab}^2 + 2m_0E_{lab}}}{E_{lab} + m_0}$$ \hspace{1cm} (2.19)

where $m_0$ is the nucleon rest mass in units of MeV/c.

The projectile energy dependence of the S-matrix is introduced through the energy dependence of the free nucleon-nucleon total cross sections. The nucleon-nucleon total cross sections are actually reduced from their free values when the reaction occurs in the nuclear medium. The effect of a density dependent nucleon-nucleon total cross sections was considered in Ref. [35] and shown to effect the reaction cross section of $^{12}$C+$^{12}$C and two-neutron removal cross sections for $^6$He and $^{11}$Li by, at most, a few percent.

The parameter $\alpha$ in Eq. 2.15 is the ratio of the real to imaginary forward scattering amplitudes and is included to account for the fact that the nucleon-nucleon amplitude is not entirely absorptive, even at high energy. This parameter is taken from fits to experimental free nucleon-nucleon cross sections and is tabulated as a function of energy in Ref. [36]. The minimum energy given by Ref. [36] is 100 MeV and for beam energies below but close to this energy we use the value for 100 MeV. In Ref. [31] the use of $\alpha$ values taken from a polynomial fit to the values of Ref. [36], extrapolated to below 100 MeV was considered and was shown to affect the diffractive cross section for $^{15}$C($-1n$) by $\approx$10%. The stripping cross sections are independent of $\alpha$ so remained unchanged.

The function $g(r)$ in Eq. 2.15 defines the profile of the nucleon-nucleon interaction and is either assumed to be a delta function (zero-range), or a finite-range Gaussian with range parameter $\beta_{NN}$. Where a finite range nucleon-nucleon interaction is used we will
2.2. CORRELATED TWO-NUCLEON KNOCKOUT

assume a range parameter $\beta_{nn} = \beta_{np} = 0.5$ fm. In the case of a zero-range interaction the integration over the beam direction $z$ components of $\vec{r}_p$ and $\vec{r}_t$ may be carried out. The $S$-matrix may then be written in terms of the projectile and target ground state densities projected onto the impact parameter plane (see Ref. [26]),

$$\rho^{(s)}(s) = \int_{-\infty}^{+\infty} dz \rho(|\vec{s} + \vec{z}|).$$

(2.20)

The density profiles for the projectile core are typically taken from spherical Hartree-Fock calculations. Those for the (stable) carbon or beryllium target are assumed to be Gaussian, with root-mean-square radius taken from reaction cross section measurements [37]. For exotic systems the Hartree-Fock calculations may suggest a proton or neutron skin - an excess of one type of nucleon near the surface of the projectile. The different density profiles of the protons and neutrons can be explicitly taken into account as was done in Ref. [3] for the case of neutron removal from $^{32}$Ar. The proton and neutron distributions for the target were assumed to have the same profile $\rho_t$, but are appropriately normalised to the number of protons or neutrons. The separate point proton and neutron density distributions for the projectile core ($\rho^p$ and $\rho^n$) can be taken directly from the Hartree-Fock calculations. The differences between the like-nucleon $t_{nn}$ and unlike-nucleon $t_{np}$ effective nucleon-nucleon interactions can then be taken into account. Explicitly, the integrand of Eq. 2.13 is rewritten as

$$t_{NN}\rho_p\rho_t \rightarrow \frac{t_{pp}\rho_t}{A_t} (Z_t\rho_p^p + N_t\rho_p^n) + \frac{t_{pn}\rho_t}{A_t} (N_t\rho_p^p + Z_t\rho_p^n).$$

(2.21)

It is assumed that $t_{nn} = t_{pp}$. The effective nucleon-nucleon interaction then contains the relevant free nucleon-nucleon total cross section and not the isospin weighted cross section of Eq. 2.15. In Ref. [3] it was demonstrated that for single-neutron knockout from neutron-deficient $^{32}$Ar, the cross section is insensitive to the (proton) skin thickness and even to the precise shape of the density distribution.

2.2 Correlated two-nucleon knockout

We consider the removal of two nucleons from an (often radioactive) $A+2$-body projectile incident on a light nuclear target, such as carbon or beryllium, at energies $>60$ MeV/nucleon. The $A$-body heavy residue is observed after the reaction. We consider population of the $A$-body residue by direct two-nucleon knockout (single-event) only. This is the only viable mechanism for well-bound nucleons, that are initially deficient in the projectile. Removal of a single well-bound nucleon to a particle-unbound state is
likely to lead to evaporation of the other type of nucleon, due to asymmetric nucleon separation thresholds. Further, if the removed nucleons are well bound the $A + 1$ body system will have a high separation threshold for nucleons of the type removed, such that the single-particle strength is exhausted before sufficiently high excitation energies are reached. Direct two-nucleon knockout is thus an efficient method of producing specific exotic species. The removal of two weakly bound nucleons populates states in nuclei less exotic than the projectile and thus is of less interest, and cases of unlike-pair removal require consideration of contributions arising from indirect (direct one-nucleon knockout followed by nucleon evaporation) two-nucleon removal. This two-step contamination of the clean, direct, one-step two-nucleon removal mechanism is discussed in some detail in Ref. [31] (see Appendix B.1).

The first experimental exploitation of direct two-nucleon removal was made in Ref. [38], which used two-proton removal from $^{36}$Si to perform $\gamma$-spectroscopy on $^{34}$Mg, extending information on the island of inversion. Whilst this study used a two-proton removal reaction as the production mechanism for the isotope of interest, it was not until later that the reaction was confirmed to proceed as a direct reaction [7], when absolute cross sections and residue momentum distributions were measured for several examples of two-proton removal and described within a direct reaction model. A fully correlated theoretical description of the absorptive part of the cross section was given in Ref. [8] and applied to several sd-shell examples. Further examples were given in Ref. [39]. The removal of two-neutrons from neutron deficient projectiles was later shown to proceed as a direct reaction [40]. The theoretical formalism was then developed to include events where one of the nucleons is removed via an elastic interaction with the target and the other via an inelastic interaction, and with simple estimates of pure elastic removal of the two nucleons. Most recently, estimates have been made for the shapes of the heavy residue longitudinal momentum distributions, suggesting that they may depend strongly on the state populated [16, 41]. The goal here is to fully develop the formalism required to calculate the residue longitudinal momentum distribution and investigate its sensitivities.

### 2.2.1 Two nucleon overlap functions

We first consider the two-nucleon knockout cross section following the formulations of Refs. [8] and [9]. To clarify the notation of these references we denote a particular final state of the heavy residue by $f$, having spin $J_f$. A particular final state $f$ in a particular magnetic substate $M_f$, is denoted by $F$, such that $F \equiv (f, M_f)$. We consider the population of a particular heavy residue state $f$; the final state exclusive cross section.
2.2. CORRELATED TWO-NUCLEON KNOCKOUT

We begin by considering the shell-model antisymmetrized two nucleon overlap, written as

\[ \Psi_{J_i M_i}^{(F)} (1, 2) \equiv \langle \Phi_{J_f M_f}^{(F)} (A) | \Psi_{J_i M_i} (A, 1, 2) \rangle = \sum_{I \mu \alpha} C_{J_i J_f}^{I \mu I} (I \mu J_f | J_i M_i) [\phi_{\beta_1} \otimes \phi_{\beta_2}]_{I \mu}. \]  \hspace{1cm} (2.22)

Here, \( \Psi_{J_i M_i} (A, 1, 2) \) is the full \((A + 2)\)-body projectile ground state wave function and \( \Phi_{J_f M_f}^{(F)} (A) \) is a particular (shell model) \( A \)-body heavy residue final state. The index \( \alpha = (n_1 \ell_1 j_1, n_2 \ell_2 j_2) \equiv (\beta_1, \beta_2) \) denotes a pair of orbitals of the two nucleons (the two-nucleon configuration), such that \( \beta_i \) denotes a given orbital in the assumed active shell-model model space. For each final state \( f \) many different orbital pairs can contribute and these are summed coherently. \( C_{\alpha}^{I \mu} \) is the signed two-nucleon amplitude which reflects the parentage and phase of the two nucleons about a particular core state in the projectile. These are typically taken from shell model calculations where they are related to reduced matrix elements of coupled nucleon creation operators \([42, 43]\).

\[ C_{\alpha}^{I \mu} = - \langle I | [a_{\beta_1}^+ \otimes a_{\beta_2}^+] | f \rangle \rangle \left( (2J_f + 1)(1 + \delta_{\beta_1, \beta_2}) \right)^{1/2}. \]  \hspace{1cm} (2.23)

Alternatively, if neglecting residual interactions in the shell model, these can be related to the appropriate two-nucleon coefficients of fractional parentage \([44]\). In this limit, the residue states are assumed to be pure two-nucleon-hole configurations, which necessitates making some (independent-particle-model) assumption about the projectile ground state wave function. For cases where all the active shells are filled, the two-nucleon amplitudes derived from coefficients of fractional parentage are simply \( \sqrt{2J_f + 1} \). Expressions for cases where the removed nucleons are from the same partially-filled shell can be found in Refs. \([8, 42, 44, 45]\).

The antisymmetrized two nucleon wave function in \( jj \)-coupling is given as

\[ [\phi_{j_1} \otimes \phi_{j_2}]_{I \mu} = D_{\alpha} \sum_{m_1 m_2} (j_1 m_1 j_2 m_2 | I \mu) \times [\phi_{j_1}^{m_1} (1) \phi_{j_2}^{m_2} (2) - \phi_{j_1}^{m_1} (2) \phi_{j_2}^{m_2} (1)]. \]  \hspace{1cm} (2.24)

where the single nucleon wave function for nucleon \( i \), \( \phi_{j}^{m} (i) \equiv \phi_{j}^{m} (\vec{r}_i) \) is given by

\[ \phi_{j}^{m} (i) = \sum_{\lambda \sigma} (\ell \lambda s \sigma | jm) \ Y_{\lambda \sigma} (\hat{r}_i) \ u_{j}(r_i) \ Y_{\lambda \sigma}, \]  \hspace{1cm} (2.25)
and the normalisation coefficient $D_\alpha$ is

$$D_\alpha = \sqrt{\frac{1}{2(1 + \delta_{\beta_1 \beta_2})}}.$$  

(2.26)

For convenience we have suppressed isospin labels in the above, since we restrict the present discussions to like-nucleon knockout. The two-nucleon amplitudes are (assumed to be) given in $pn$-formalism where the proton and neutron orbitals are treated independently. The use of isospin TNA requires (i) multiplication of Eq. 2.22 by the appropriate isospin Clebech-Gordan coefficient $(T_\tau T_f T_i | T_\tau i)$, (ii) the addition of the phase $(-1)^{T+1}$ to the second (exchange) term in the square bracket of Eq. 2.24 and (iii) multiplication of the same equation by an isospin wave function $\chi_{T_\tau}(1, 2)$. Additional isospin labels and summations are required. We consider isospin explicitly when discussing $LS$-coupling. The isospin equivalents of Eqs. 2.22, 2.24 and 2.25 can be found in Section 2.3.5. Since protons and neutron are distinguishable, if unlike-pair knockout were described using $pn$-formalism two-nucleon amplitudes, the two-nucleon wave function would be a simple product of single-nucleon wave functions. Additional two-nucleon amplitudes are then required to explicitly take into account the interchange of proton and neutron orbitals.

The angular momentum coupling of the two nucleons is shown schematically in Fig. 2.2. The formalism presented here is developed in $jj$-coupling, where the nucleon orbital angular momentum $\ell$ and intrinsic spin $s$ are coupled to total angular momentum $j$, whereupon the two nucleons are coupled $\vec{j}_1 + \vec{j}_2 = \vec{J}$. Later (Section 4.2) we shall consider $LS$-coupling, where the nucleon orbital angular momenta $\ell_1$ and $\ell_2$ are first coupled to total $L$, with the intrinsic spins coupled to $S$. This proves useful when considering the residue momentum distribution. The $LS$-coupling scheme was also used to assess the $S = 0$ and $S = 1$ content of the two-nucleon overlap function [8].

The eikonal reaction dynamics restrict the projectile constituents to follow straight line paths past the target, so that each constituent’s impact parameter $b_i$ is fixed. The projectile core-valence nucleon co-ordinates are naturally written in cylindrical polar form $\vec{r} \equiv (s, \varphi, z)$, where the spherical polar angle $\theta$, satisfies $\cos \theta = z/r$. These are shown in the impact parameter $(x, y)$ plane in Fig. 2.3. We assume that, since the projectile core is significantly more massive than the two nucleons, the core impact parameter and the center-of-mass impact parameter are coincident, dubbed the no-recoil approximation.
2.2. CORRELATED TWO-NUCLEON KNOCKOUT

\[
\begin{align*}
&[LS] I \\
&[j_1 j_2] I \\
&[I_1 j_1] L \\
&[I_2 s j_2] J_f, T_f
\end{align*}
\]

Figure 2.2: Angular momentum coupling schemes used in the calculations. The initial projectile ground state and final residue state have spin \( J_i (M_i) \) and \( J_f (M_f) \) and isospins \( T_i (\tau_i) \) and \( T_f (\tau_f) \) respectively. Each two nucleon configuration \( \alpha \) consists of two orbitals \( \beta_1 \) and \( \beta_2 \) of spherical shell-model quantum numbers \( n_i, \ell_i, j_i (\lambda_i \text{ and } m_i) \). In \( jj \)-coupling the total angular momentum of the two nucleons \( j_1 + j_2 \) is denoted by \( I (\mu) \), which must also couple the initial and final state spins. In \( LS \) coupling, we couple the orbital angular momenta \( \ell_1 \) and \( \ell_2 \) to \( L (\Lambda) \), the intrinsic spins to \( S (\Sigma) \) and then \( L \) and \( S \) to the total angular momentum \( I \), which again must couple the initial and final state spins. The projections of the angular momenta described above are shown in brackets.

\[
\begin{align*}
&\hat{b} \\
&\hat{b}_i \\
&\hat{b}_1 \\
&\hat{b}_2 \\
&\hat{s}_1 \\
&\hat{s}_2
\end{align*}
\]

Figure 2.3: Co-ordinates used in the calculations in the impact parameter plane. The core-valence nucleon co-ordinate \( \vec{r}_i \) has projection on the impact parameter plane (that perpendicular to the beam direction) of \( \hat{s}_i \equiv (s_i, \varphi_i) \). The \( \hat{x} \) direction is defined as the direction of the center-of-mass impact parameter \( \hat{b} \), such that \( \varphi_i \) is the angle between \( \hat{b} \) and \( \hat{s}_i \). The impact parameters of the nucleons are \( \vec{b}_i = \vec{b} + \hat{s}_i \), where \( \vec{b} \) is the centre-of-mass impact parameter, which under the no-recoil approximation is coincident with the heavy residue impact parameter.
2.2. CORRELATED TWO-NUCLEON KNOCKOUT

2.2.2 Absorption cross section

Our starting point, as in Refs. [8] and [9], is the absorption cross section for a composite three-body projectile, consisting of two valence nucleons 1 and 2, and a heavy core. The absorption cross section, where the interaction leaves the target in a state other than its ground state, is written as

$$\sigma_{abs} = \frac{1}{\hat{J}_i^2} \sum_{M_i} \int d\vec{b} \langle \Psi_{J_iM_i} | 1 - |S_f|^2 |S_1|^2 |S_2|^2 | \Psi_{J_iM_i} \rangle , \quad (2.27)$$

where $$\Psi_{J_iM_i}$$ is the three body projectile ground state wave function. The elastic scattering S-matrices are denoted by $$S_i$$ for nucleons 1 and 2 and the final residue f, calculated as described in Section 2.1.3. They are each a function of the respective impact parameters $$b_i = |\vec{b} + \vec{s}_i|$$. The abbreviation $$\hat{J}_i = \sqrt{2J_i + 1}$$ is used extensively throughout. The summation over $$M_i$$ arises from the average over the projectile ground state spin projections.

When considering two-nucleon removal reactions we are interested in events where the two-nucleons are removed from the projectile and the heavy core (or residue) is measured after the reaction, having not suffered an absorptive collision with the target. We extract these events from the absorption cross section by expanding the absorption operator using

$$1 = \prod_{i=c,1,2} |S_i|^2 + (1 - |S_i|^2)$$

and retain only terms where the core is elastically scattered during the reaction, i.e. those containing the core survival probability $$|S_f|^2$$. The remaining contributions to the absorption cross section involve the operator

$$O_{ko} = |S_f|^2 \left[ (1 - |S_1|^2)(1 - |S_2|^2) + |S_1|^2(1 - |S_2|^2) + (1 - |S_1|^2)|S_2|^2 \right] . \quad (2.28)$$

These terms reflect the underlying two-nucleon removal mechanism, where the residue final states interact elastically with the target. The first term describes events where both nucleons are absorbed by the target and the second two terms describe events where one of the two nucleons is elastically scattered and the other absorbed.

The stripping (inelastic breakup) cross section was formulated and discussed in Ref. [8] and arises from the first term in the square bracket of Eq. 2.28 involving $$(1 - |S_1|^2)(1 - |S_2|^2)$$, i.e.

$$\sigma_{str} = \frac{1}{\hat{J}_i^2} \sum_{M_i} \int d\vec{b} \langle \Psi_{J_iM_i} ||S_f|^2 (1 - |S_1|^2)(1 - |S_2|^2) | \Psi_{J_iM_i} \rangle . \quad (2.29)$$

In addition to the purely absorptive terms, $$O_{ko}$$ contains terms where one of the two nucleons, 1 say, undergoes an elastic collision, described by $$|S_1|^2$$, whereas the other, 2, is absorbed by the target. These events where nucleon 1 interacts elastically correspond to
the cross section $\sigma_1$, given as,

$$\sigma_1 = \frac{1}{J_i^2} \sum_{M_i} \frac{1}{M_i} \int d\vec{b} \langle \Psi_{J_i M_i} | | S_f |^2 | S_1 |^2 (1 - |S_2|^2) | \Psi_{J_i M_i} \rangle .$$  \hspace{1cm} (2.30)$$

There will also be cross section corresponding to the elastic scattering of nucleon 2 and the absorption of nucleon 1, denoted $\sigma_2$.

We assume that the residue-target $S$-matrix is diagonal with respect to final states of the heavy residue $F$, such that the reaction induces no dynamical excitation of the core. We also assume that the residue-target $S$-matrix for each of the residue excited states $f$ is the same as the ground state, denoted $S_c$. This approximation is termed the “spectator-core” approximation in the literature. Having also made the spectator-core and no-recoil approximations we can remove explicit dependence on the residue final-state co-ordinates and write,

$$\langle \Phi_F^{(F)} | | S_f |^2 | \Phi_{F'}^{(F')} \rangle = \delta_{F, F'} (|S_c(b)|^2) \equiv \delta_{f, f'} \delta_{M_f, M_{f'}} (|S_c(b)|^2) .$$  \hspace{1cm} (2.31)$$

The spectator-core approximation gives an incoherent sum over the final residue states $f$. Further, it will allow us to sum over the projections of the projectile ground state $M_i$ and residue final state $M_f$, a consequence of which is that the cross section is an incoherent sum of the contributing nucleon pair total angular momenta $I$. The no-recoil approximation allows us to remove the residue final state (core) $S$-matrix from the integral over the two-nucleon co-ordinates. The part of the total absorption cross section corresponding to events where the core survives the interaction is then given by

$$\sigma_{abs}^{(c)} = \sigma_{str} + \sigma_1 + \sigma_2 = \sum_f \left[ \frac{1}{J_i^2} \sum_{M_i} \frac{1}{M_i} \int d\vec{b} |S_c|^2 \langle \Psi_{J_i M_i} | (1 - |S_1|^2)(1 - |S_2|^2) 
+ |S_1|^2(1 - |S_2|^2) + (1 - |S_1|^2)|S_2|^2 \langle \Psi_{J_i M_i} \rangle \right] ,$$  \hspace{1cm} (2.32)$$

where, instead of the three-body wave function $\Psi_{J_i M_i}(A + 2)$, we now have the two-nucleon overlap function $\Psi_{J_i M_i}^{(F)}(1, 2)$ and $\langle ... \rangle$ indicates integration over spin and spatial co-ordinates of the two nucleons. Note that this is not the same as the two-nucleon removal cross section. The first term in the square bracket corresponds to two-nucleon removal via pure absorption $\sigma_{str}$, but for the second two terms there is no guarantee that the elastically scattered nucleon is not bound to the target following the reaction. In fact, the latter two-terms are much closer to single-nucleon stripping than two-nucleon knockout.
2.2.3 Reaction mechanisms

The expanded absorption cross section Eq. 2.32 contains terms where both nucleons interact inelastically with the target \((1 - |S_i|^2)(1 - |S_j|^2)\), described in Ref. [8], and events where one interacts inelastically and the other elastically \((1 - |S_i|^2)|S_j|^2\), described in Ref. [9]. The total operator for events where the core is not absorbed in the reaction is written as

\[
O_{ko}(c, 1, 2) = O_{str}(c, 1, 2) + O_{ds}(c, 1, 2), \tag{2.33}
\]

where

\[
O_{str}(c, 1, 2) = |S_c|^2 K_1(1, 2) = |S_c|^2 (1 - |S_1|^2)(1 - |S_2|^2), \tag{2.34}
\]

\[
O_{ds}(c, 1, 2) = |S_c|^2 K_2(1, 2) = |S_c|^2 \left[ (1 - |S_1|^2)|S_2|^2 + |S_1|^2 (1 - |S_2|^2) \right]. \tag{2.35}
\]

The term \(O_{str}(c, 1, 2)\), referred to as pure stripping or simple stripping, describes events where both nucleons interact violently with the target such that the nucleons are no longer bound to the core and the target does not remain in its ground state. The term \(O_{ds}(c, 1, 2)\) referred to as diffractive-stripping describes events where one nucleon is absorbed by the target and the second is elastically scattered.

The operator \(|S_i|^2\) describes the survival probability of the valence nucleon and target. The overlap integral \(\langle \Psi^{(F)}_{J, M_i} | (1 - |S_i|^2)|S_j|^2 |\Psi^{(F)}_{J, M_i} \rangle\) implicitly includes many events where the valence nucleon interacts elastically with the target but remains bound to the core of the projectile, and many more events where the valence nucleon does not interact with the target at all. Such events do not populate \(A\)-body residue states. In order to remove contributions where the valence nucleons remain bound we project off the bound states of the valence nucleon \(|\phi_{m''}^{j''}\rangle\) by rewriting

\[
|S_i|^2 \to S_i^* \left[ 1 - \sum_{j'' m''} |\phi_{m''}^{j''}\rangle \langle \phi_{m''}^{j''}| \right] S_i, \tag{2.36}
\]

for the diffracted nucleon. The sum over bound states \(j''\) extends over the active orbitals, consistent with the shell-model model space. This generates the original \(K_2(1, 2)\) from the first term in the square bracket and an additional term \(K_3(1, 2)\) from the second (projection) term,

\[
O_{ds}(c, 1, 2) = |S_c|^2 [K_2(1, 2) - K_3(1, 2)], \tag{2.37}
\]
2.2. CORRELATED TWO-NUCLEON KNOCKOUT

where $K_3(1, 2)$ is given as

$$K_3(1, 2) = \sum_{j' m''} \left[ S_1^* |\phi_j^{m''} \rangle (\phi_j^{m''} | S_1 (1 - |S_2|^2) ight.$$  

$$+ (1 - |S_1|^2) S_2^* |\phi_j^{m''} \rangle (\phi_j^{m''} | S_2) \right].$$

(2.38)

The cancellation between the terms $K_2(1, 2)$ and $K_3(1, 2)$ may necessarily be very large since the largest contributions to $K_2(1, 2)$ arise when the diffracted nucleon is far from the target and $|S_i|^2 \to 1$. At large impact parameters, the diffracted nucleon is expected to interact weakly with the target and remain bound to the core. Later, we develop the formalism for residue longitudinal momentum distributions in terms of the pure stripping events, and subsequently discuss the diffractive-stripping terms in Sections 2.2.6 and 2.3.6.

2.2.4 Surface localisation wave function sampling

Our outline picture of the reaction has the projectile undergoing a grazing collision with the target, during which two nucleons from the projectile are removed by the stripping and diffraction mechanisms. The two nucleons are rather like the juvenile porridge thief Goldilocks; if too near to the centre of the projectile, the projectile core will be absorbed by the target; if too far away from the core, the probability for finding the nucleons, determined by the single nucleon wave functions, is too small. This picture requires the two nucleons to be near the surface of the projectile for them to be removed in the reaction. This spatial localisation of the reaction means that, regardless of the mechanism of removal, be it elastic or inelastic, the removed nucleons must originate from a similar volume of the projectile ground state. The momentum content of the wave function in this sampled volume will then be relatively independent of the details of the reaction mechanism.

Given that the core is a spectator, that the $S$-matrix is independent of the final state $f$ and is diagonal with respect to the final core states, we can integrate out explicit dependence on the internal degrees of freedom of the core. That is, the $A$ nucleon coordinates of the projectile core can be eliminated, giving

$$\langle \Psi_{J_i M_i} | \Psi_{J_i M_i} \rangle_A = \sum_f \left[ \sum_{M_f} \left| \Psi^{(F)}_{J_i M_i, (1, 2)} \right|^2 \right].$$

(2.39)

Using the expression for the two nucleon overlap we can write the knockout cross section,
populating a particular heavy residue state \( f \), as
\[
\sigma^{(f)}_{ko} = \frac{1}{J_i^2} \sum_{M_i, M_f} \int d\vec{b} \langle \Psi_{J_i M_i}^{(F)} \mid \mathcal{O}_{ko} \mid \Psi_{J_i M_i}^{(F)} \rangle.
\] (2.40)

Since \( \mathcal{O}_{ko} \) depends only on the centre of mass impact parameter \( \vec{b} \) and impact parameter plane projections \( \vec{s}_1 \) and \( \vec{s}_2 \), if the integrals over \( \vec{r}_i \) are written in cylindrical co-ordinates \( \vec{r}_i = (\vec{s}_i, z_i) \), we can take \( \mathcal{O}_{ko} \) outside the integrals over \( z_i \), to give
\[
\sigma^{(f)}_{ko} = \int d\vec{b} \int d\vec{s}_1 \int d\vec{s}_2 \mathcal{P}_f(\vec{s}_1, \vec{s}_2) \mathcal{O}_{ko}(\vec{b}, \vec{s}_1, \vec{s}_2).
\] (2.41)

Here, we have defined the joint position probability \( \mathcal{P}_f(\vec{s}_1, \vec{s}_2) \), given by \cite{17},
\[
\mathcal{P}_f(\vec{s}_1, \vec{s}_2) = \frac{1}{J_i^2} \sum_{M_i, M_f} \int dz_1 \int dz_2 \langle \mid \Psi_{J_i M_i}^{(F)} (1, 2) \rangle^2 \rangle_{sp},
\] (2.42)
where \( \langle \ldots \rangle_{sp} \) indicates integration over the spin coordinates. Here, and elsewhere in the following, the integrals over \( z_i \) are taken to be over the range \(-\infty \) to \(+\infty \). Essentially, the knockout reaction probes the properties of the projectile ground state wave function in a volume defined by the \( S \)-matrices, necessarily near the surface of the projectile.

The surface localisation of the stripping cross section is demonstrated by considering the differential cross section with respect to the projection of a core-nucleon vector \( (\vec{r}_1) \) on the impact parameter plane \( s_1 \). This is shown in Fig. 2.4 for the case of two-proton removal from \(^{28}\text{Mg}\). The two-protons are bound by half the two-proton separation energy \( S_{2p}(^{28}\text{Mg})/2 = 15 \) MeV. The precise details of the radial wave functions can be found in later Sections. It is compared to the projected density of the projectile core \( \rho^{(z)}(s) \) given by Eq. 2.20. The events contributing to two-nucleon removal are expected to localise the two nucleons to a region between the centres of the projectile core and the target. This localisation of \( \vec{s}_1 \) makes \( \rho^{(z)}(s) \) the best measure of the local core density, though it is normalised such that
\[
A = 2\pi \int_0^\infty ds \ s \ \rho^{(z)}(s),
\] (2.43)
where \( A \) is the mass of the projectile core. The peak cross section for this example is localised at \( s \approx 3.5 \) fm, where the core projected density has fallen to approximately 20% of the central value. Though the nucleons are strongly bound, the reaction will certainly probe significantly more than the tail of the nucleon wave function; two-nucleon knockout events occur for \( s \approx 2 - 6 \) fm for \(^{28}\text{Mg}(-2p)\), where the root-mean-square radius of the
Figure 2.4: Surface localisation of the stripping cross section for $[0d_{5/2}]^2$ removal from $^{28}\text{Mg}$ for $I = 0$ (solid), 2 (dotted) and 4 (dashed). The projected core density profile (or thickness function) $\rho_z(s)$ is shown by the solid circles. The cross sections are isolated at the projectile surface and is relatively insensitive to the coupling of the two nucleons. Note the normalisation of the curves is somewhat different. Integrating $d\sigma/ds$ over $s$ will give the stripping cross section $\sigma^{(f)}_{\text{str}}$. To obtain the total number of core nucleons we must integrate $\rho_z(s)$ over $\vec{s}$ i.e. vectorial $s$, essentially integrating $s\rho_z(s)$. The reason we have plotted $\rho_z(s)$ and not $2\pi s\rho_z(s)$ is that the cross section is expected to arise from a relatively small region between the projectile and target i.e. a narrow band of the azimuthal angle $\varphi$, where $\rho_z(s)$ is a more reasonable measure of the local core density.
radial wave function is \( \approx 3.3 \, \text{fm} \).

### 2.2.5 Simplification of the stripping terms

Whilst the expression (Eq. 2.41) of the cross section in terms of the two-nucleon joint position probability is useful, for computational reasons we need to factorise the multidimensional integrals as far as possible. In the following we will consider only the pure stripping cross section, resulting from events where both nucleons are absorbed by the target and the heavy core of the projectile survives the collision. Calculation of the diffractive-stripping term follows in a near-identical manner, by replacing the stripping operator \( O_{\text{str}}(1, 2) \) with the diffractive-stripping operator \( O_{\text{ds}}(1, 2) \). Having made the above expansion of the absorption cross section (Eq. 2.32) we can write the total two nucleon stripping (absorption) cross section \( \sigma_{\text{str}} \) as a sum of the final-state-exclusive stripping cross sections \( \sigma_{\text{str}}^{(f)} \),

\[
\sigma_{\text{str}} = \sum_{f} \sigma_{\text{str}}^{(f)}.
\]  

(2.44)

Here the final-state exclusive cross section is

\[
\sigma_{\text{str}}^{(f)} = \frac{1}{J_i^2} \sum_{M_i, M_f} \int d\vec{b} \, |S_c|^2 \langle \Psi_{J_i M_i}^{(F)} | K_1(1, 2) | \Psi_{J_i M_i}^{(F)} \rangle,
\]  

(2.45)

where the bra-ket \( \langle ... \rangle \) denotes integration over the spin and valence nucleon co-ordinates \( \vec{r}_i \). We extract explicitly the integration over the nucleon co-ordinates \( \vec{r}_i \),

\[
\langle \Psi_{J_i M_i}^{(F)} | K_1(1, 2) | \Psi_{J_i M_i}^{(F)} \rangle = \int d\vec{r}_1 \int d\vec{r}_2 \, K_1(1, 2) \langle \Psi_{J_i M_i}^{(F)} | \Psi_{J_i M_i}^{(F)} \rangle_{\text{sp}},
\]  

(2.46)

after which \( \langle ... \rangle_{\text{sp}} \) denotes the spin-integrated two-nucleon overlap. We now consider this further. By inserting the anti-symmetrized two nucleon wave function, Eq. 2.24, we obtain

\[
\frac{1}{J_i^2} \sum_{M_i, M_f} \langle \Psi_{J_i M_i}^{(F)} | \Psi_{J_i M_i}^{(F)} \rangle_{\text{sp}} = \frac{1}{J_i^2} \sum_{I \mu \mu' \alpha \alpha' M_i, M_f} C_{\alpha J_i J_i}^I C_{\alpha' J_i J_i}^{I'} (I \mu J_f M_f | J_i M_i) (I' \mu' J_f M_f | J_i M_i)
\]

\[
\langle [\phi_{j_1}(1) \otimes \phi_{j_2}(2)]_{I \mu}^{|\phi_{j_1}(1) \otimes \phi_{j_2}(2)}_{I \mu'} \rangle_{\text{sp}}.
\]  

(2.47)

The spectator-core approximations prohibits dynamic excitation of the core, such that \( F = F' \) (Eq. 2.31). This results in an incoherent sum over the final state spin-projections.
\[ M_f. \text{ We can now sum over these projections using,} \]
\[
\sum_{M_i M_f} (I \mu J_f M_f | J_i M_i) (I' \mu' J_f M_f | J_i M_i) = \frac{j_f^2}{f^2} \delta_{I,I'} \delta_{\mu, \mu'}
\]  
(2.48)
giving also an incoherent sum over \( I \) and \( \mu \). Inserting this result into Eq. 2.47 gives
\[
\frac{1}{j_i^2} \sum_{M_i M_f} \langle \Psi^{(F)}_{J_i M_i} | \Psi^{(F)}_{J_i M_i} \rangle_{sp} = \sum_{\lambda \alpha \alpha'} C^i_{\alpha} C^{i'}_{\alpha'} D_{\alpha} D_{\alpha'}
\]
\[
\sum_{m_1 m_2 m'_1 m'_2} (j_1 m_1 j_2 m_2 | I \mu) (j'_1 m'_1 j'_2 m'_2 | I \mu)
\]
\[
\left[ (\phi_{j_1}^{m_1} | \phi_{j_2}^{m_2})_{sp} (\phi_{j'_1}^{m'_1} | \phi_{j'_2}^{m'_2})_{sp} + (\phi_{j_2}^{m_2} | \phi_{j_2}^{m_2})_{sp} (\phi_{j'_1}^{m_1} | \phi_{j'_1}^{m_1})_{sp}
\right.
\]
\[
\left. - (\phi_{j_1}^{m_1} | \phi_{j_2}^{m_2})_{sp} (\phi_{j'_2}^{m'_2} | \phi_{j'_1}^{m_1})_{sp} - (\phi_{j_2}^{m_2} | \phi_{j_2}^{m_2})_{sp} (\phi_{j'_1}^{m_1} | \phi_{j'_2}^{m'_2})_{sp} \right].
\]
(2.49)

We have introduced the spin-integrated alternate braket \((\phi_j^{m'} | \phi_j^{m})_{sp}\) for an individual nucleon, written, as in Ref. [8], as
\[
(\phi_j^{m'} | \phi_j^{m})_{sp} = \sum_{\lambda \lambda' \sigma} (\ell \lambda s \sigma | j m) (\ell' \lambda' s \sigma | j' m') u_{j' \ell'} (r) u_{j \ell} (r) Y_{\lambda} (\hat{r}) Y_{\lambda'} (\hat{r})^* 
\]
\[
= \sum_{kq} (j' m' k q | j m) \left[ (-1)^{2s+j+j'-\ell} \frac{\hat{l} \hat{l}'}{\sqrt{4\pi}} (\ell \ell' | k 0) \right.
\]
\[
\left. W (j s k \ell'; j' \ell') u_{j' \ell'} (r) u_{j \ell} (r) Y_{kj} (\hat{r}) \right]
\]
\[
= \sum_{kq} (j' m' k q | j m) Z_{kj}^{\beta \gamma}(\hat{r}),
\]
(2.50)
where in the second line we have used the addition theorem to combine the spherical harmonics and summed over the projections \( \lambda, \lambda' \) and \( \sigma \). \( Z_{kj}^{\beta \gamma}(\hat{r}) \) is a shorthand for the square bracketed expression where we recall \( \beta \) indicates a particular orbital. Previously (Refs. [8] and [9]) this was written as \( \langle j' \ell' | \mathcal{O}_{kj}(\hat{r}) | j \ell \rangle \).

Note that the four (square bracketed) terms in Eq. 2.49 can be reduced to two for calculation of the cross section. The first (third) and second (fourth) terms are identical save for the interchange of the co-ordinates of nucleons 1 and 2, and as the operator \( \mathcal{S}_i \) is symmetric with respect to interchange of 1 and 2, the two terms are equivalent.

Equation 2.49 contains four products of two brakets \((\phi_j^{m'} | \phi_j^{m})_{sp}\), the first and second brakets of each pair being for nucleons 1 and 2 respectively, which arise from the product of anti-symmetrized two-nucleon wave functions. The first two products of brakets are
referred to as *direct* terms and the last two as *exchange* terms. In calculating the stripping cross section one must evaluate the product of two such brakets as shown in Eq. 2.49 and one can recouple the relevant angular momentum as was done in Ref. [8]. For the purposes of calculating residue momentum distributions it is preferable to retain the dependence on the projection of the orbital angular momenta $\lambda$, so we simply write $(\phi_{j'i'}^m | \phi_j^m)$ as

$$
(\phi_{j'i'}^m | \phi_j^m)_{sp} = \sum_{\lambda' \lambda \sigma} (\ell \lambda \sigma | jm) (\ell' \lambda' \sigma | j'm') \exp[i(\lambda - \lambda')\varphi]
$$

where the coefficients $C_{\ell \lambda}$ arise from the spherical harmonics and are defined [46] as

$$
C_{\ell \lambda} = \frac{1}{\sqrt{4\pi}} \left( \frac{(2\ell + 1)(\ell - \lambda)!}{(\ell + \lambda)!} \right)^{1/2}, \quad \lambda \geq 0, \quad (2.51)
$$

$$
C_{\ell \lambda} = (-1)^\lambda C_{\ell |\lambda|}, \quad \lambda < 0. \quad (2.52)
$$

For completeness, we show the recoupling of the stripping cross section for cases where only the momentum-integrated cross section is required. We first integrate the alternate braket $(\phi_{j'i'}^m | \phi_j^m)_{sp}$ over the nucleon position co-ordinates, incorporating the relevant $S$-matrix, giving

$$
(\phi_{j'i'}^m | (1 - |S_i|^2) | \phi_j^m) = \int d\vec{r}_i (\phi_{j'i'}^m | (1 - |S_i|^2) | \phi_j^m)_{sp},
$$

$$
= \sum_{kq} (j'm'kq|jm) \int d\vec{r}_i (1 - |S_i|^2) Z_{kq}^{j'm'}(\vec{r}_i)
$$

$$
= \sum_{kq} (j'm'kq|jm) Z_{kq}^{j'm'}(b), \quad (2.54)
$$

where we have defined the integral over the co-ordinates of nucleon $i$, $Z_{kq}^{j'm'}(b)$,

$$
Z_{kq}^{j'm'}(b) = \int d\vec{s}_i (1 - |S_i(b_i)|^2) \int dz_i Z_{kq}^{j'm'}(\vec{r}_i). \quad (2.55)
$$

Previously the function $Z_{kq}^{j'm'}(b)$ was denoted by \{\text{j'+}F_{kq}(b)\text{j} \text{f}\}. It remains to sum over the projections $m_1$, $m_2$, $m_1'$ and $m_2'$ for the product of two brackets,

$$
\sum_{m_1m_2m_1'm_2'\mu} (j_1'm_1'k_1q_1|j_1m_1) (j_2'm_2'k_2q_2|j_2m_2) (j_1'm_1'j_2'm_2'|I\mu) (j_1m_1j_2m_2|I\mu)
$$

$$
= (-1)^{q_1} (-1)^{l_j-j_1-j_2} \hat{Z}_{k_1}^{j_1j_2} \hat{W}(j_1j_2j_2'; k_1 I) \delta_{k_1, k_2} \delta_{q_1, -q_2}. \quad (2.56)
$$
For simplicity and consistency with Refs. [8] and [9], we make the substitutions \( k_1 = k_2 = K \) and \( q_1 = -q_2 = Q \). The second two terms in Eq. 2.49 are given by the same form with the additional phase \((-1)^{j_1 + j_2 - I}\) arising from the rearrangement of the Clebsch-Gordan coefficient \( (j_1m_1j_2m_2|I\mu) \) from the two-nucleon overlap. As noted previously, we need only consider two such products.

Substituting Eqs. 2.54 and 2.56 into Eq. 2.49 gives,

\[
\frac{1}{J_i^2} \sum_{M_iM_f} \langle \Psi_{J_iM_i}^{(F)} | K_1(1, 2) | \Psi_{J_iM_i}^{(F)} \rangle = \sum_{I\alpha\alpha'} 2 C_{\alpha\alpha'}^{J_iJ_f} C_{\alpha'\alpha}^{J_fJ_f} D_{\alpha} D_{\alpha'} \hat{j}_1 \hat{j}_2 \sum_{KQ} \frac{(-1)^Q}{K^2} \left[ \text{direct} - \text{exchange} \right],
\]

where the \textit{direct} and \textit{exchange} terms are given by,

\[
direct = (-1)^{I-j_1-j_2} W(j_1 j_1' j_2 j_2'; KI) \ Z_{KQ}^{\beta_1\beta_1'}(b) \ Z_{K-Q}^{\beta_2\beta_2'}(b),
\]

\[
exchange = (-1)^{j_2-j_1} W(j_2 j_1' j_1 j_2'; KI) \ Z_{KQ}^{\beta_2\beta_1'}(b) \ Z_{K-Q}^{\beta_1\beta_2'}(b).
\]

The two-nucleon stripping cross section, Eq. 2.45, is obtained by integrating the Eq. 2.57 over all center of mass impact parameters,

\[
\sigma_{st}^{(f)} = \frac{1}{J_i^2} \sum_{M_iM_f} \int d\vec{b} |S_c|^2 \langle \Psi_{J_iM_i}^{(F)} | K_1(1, 2) | \Psi_{J_iM_i}^{(F)} \rangle \ Z_{KQ}^{\beta_1\beta_1'}(b) \ Z_{K-Q}^{\beta_2\beta_2'}(b).
\]

\[
\frac{1}{J_i^2} \sum_{M_iM_f} \int d\vec{b} |S_c|^2 \langle \Psi_{J_iM_i}^{(F)} | K_1(1, 2) | \Psi_{J_iM_i}^{(F)} \rangle \ Z_{KQ}^{\beta_1\beta_1'}(b) \ Z_{K-Q}^{\beta_2\beta_2'}(b) \sum_{KQ} \frac{(-1)^Q}{K^2} \left[ \text{direct} - \text{exchange} \right].
\]

2.2.6 Diffractive-stripping cross section

We will now consider the diffractive-stripping cross section, which describes events where one-nucleon is absorbed by the target and the other dissociated elastically, such that it is no longer bound to the projectile. Such events are part of the total absorption cross section Eq. 2.27, but we must be careful to project-off the nucleon bound states to ensure the diffracted nucleon does not remain bound.

It was noted in Section 2.2.3 that for \( \sigma_{ds} \) as written, we expect the cancellation between the diffractive-stripping terms \( \mathcal{K}_2 \) and \( \mathcal{K}_3 \) to be very large, since there are large contributions to both terms when the non-absorbed (diffracted) nucleon is far from the target, where \( S_i(b_i) \to 1 \). A large proportion of contributions from \( \mathcal{K}_2 \) will result in the
elastically scattered nucleon remaining bound to the core, corresponding to single-nucleon stripping events, which are not of interest here. In fact the diffractive dissociation events require a relatively strong interaction with the target in order to remove the nucleon from the core, involving smaller impact parameters. Under the no-recoil approximation it is possible to reduce the cancellation by rewriting the diffraction part of $K_{ds}$. Consider the braket introduced in Eq. 2.49 and described by Eq. 2.54. For the diffractive-stripping cross section we must evaluate brakets of the form,

$$ (\phi_{j'}^{m'} | [K_{2}^{(i)} - K_{3}^{(i)}] | \phi_{j}^{m} ) = (\phi_{j'}^{m'} || S_{i} || \phi_{j}^{m} ) - \sum_{j''m''} (\phi_{j'}^{m'} | S_{i}^{*} | \phi_{j''}^{m''} )(\phi_{j''}^{m''} | S_{i} | \phi_{j}^{m} ) $$  \hspace{1cm} (2.61)

where we have used $K_{2}^{(i)}$ to refer to the part of $K_{2}$ that refers to the diffracted nucleon $i$, i.e. $|S_{i}|^{2}$. There is, of course, the stripping of the second nucleon to be considered, but since we have made the no-recoil approximation the $S$-matrices for each nucleon only depend on the centre of mass (core) impact parameter and the relevant valence nucleon co-ordinate. The stripped nucleon need not be considered here. Taking Eq. 2.61, we make the substitution $S_{i} = 1 - (1 - S_{i})$ in $K_{2} - K_{3}$ for the diffracted nucleon, to give

$$ (\phi_{j'}^{m'} || [1 - (1 - S_{i})]^{2} | \phi_{j}^{m} ) - \sum_{j''m''} (\phi_{j'}^{m'} || [1 - (1 - S_{i})]^{*} | \phi_{j''}^{m''} )(\phi_{j''}^{m''} || [1 - (1 - S_{i})] | \phi_{j}^{m} ) $$

$$ = (\phi_{j'}^{m'} || 1 - S_{i}^{2} | \phi_{j}^{m} ) - \sum_{j''m''} (\phi_{j'}^{m'} || (1 - S_{i})^{*} | \phi_{j''}^{m''} )(\phi_{j''}^{m''} || (1 - S_{i}) | \phi_{j}^{m} ). $$  \hspace{1cm} (2.62)

The second line arises through expansion of operators and the orthogonality of the bound state wave functions. The sum over bound states $j''$ is taken over all active orbitals, consistent with the shell-model model space. Note that this substitution is only possible under the no-recoil approximation, where the core $S$-matrix, and indeed stripped nucleon $S$-matrix, are not dependent of the co-ordinates of the diffracted nucleon. In effect we can rewrite the diffraction operators as

$$ K_{2}(1, 2) \rightarrow \tilde{K}_{2}(1, 2) = |1 - S_{1}|^{2}(1 - |S_{2}|^{2}) + (1 - |S_{1}|^{2})|1 - S_{2}|^{2}, $$

$$ K_{3}(1, 2) \rightarrow \tilde{K}_{3}(1, 2) = \sum_{j''m''} \left[ (1 - S_{1})^{*} | \phi_{j''}^{m''} \rangle \langle \phi_{j''}^{m''} | (1 - S_{1})(1 - |S_{2}|^{2}) 

+ (1 - |S_{1}|^{2})(1 - S_{2})^{*} | \phi_{j''}^{m''} \rangle \langle \phi_{j''}^{m''} | (1 - S_{2}) \right]. $$  \hspace{1cm} (2.63)

Whilst the difference of the operators $\tilde{K}_{2} - \tilde{K}_{3}$ is identical to $K_{2} - K_{3}$ (Eq. 2.62), it is clear that $K_{2}$ and $\tilde{K}_{2}$ are different.

This (effective) simple replacement $S_{i} \rightarrow 1 - S_{i}$ emphasizes the surface localisation of the cross section and greatly reduces the significance of the projection-off term $K_{3}$.
similar substitution was made in Ref. [47] for single-nucleon removal by diffraction, the justification being better integral convergence. Indeed, such a substitution is even more significant in single-nucleon removal since in the present case, the requirement for the other nucleon to be stripped places some limit on the integral over $b$. We will consider the residue momentum distributions associated with $K_2$ and $\bar{K}_2$ in Sections 2.3.6 and 6.4.

The nature of the diffractive-stripping operator is more complicated than that for pure-stripping. We obtain one term, identical in structure to the pure-stripping term, and a second which contains the projection-off part. We require brackets similar to those of Eq. 2.54 for each diffraction operator $\bar{K}_2$ and $\bar{K}_3$. For the $\bar{K}_2$ we need

\[
(\phi_{j'}^{m'}|1 - S_i|^2|\phi_j^m) = \int d\vec{r}_i (\phi_{j'}^{m'}|1 - S_i|^2|\phi_j^m)_{sp}
\]

\[
= \sum_{kq} (j'm'kq|jm) \int d\vec{r}_i |1 - S_i|^2 Z_{kq}^{j'\beta'}(\vec{r}_i)
\]

\[
= \sum_{kq} (j'm'kq|jm) A_{kq}^{\beta\beta'}(b),
\]

where we have defined

\[
A_{kq}^{\beta\beta'}(b) = \int d\vec{s}_i |1 - S_i|^2 \int d\vec{z}_i Z_{kq}^{j'\beta'}(\vec{r}_i).
\]

$A_{kq}^{\beta\beta'}(b)$ is similar in structure to $Z_{kq}^{j'\beta'}(b)$ and corresponds to the function $\{j'\beta'|H_{KQ}(b)|j\beta\}$ of [9], though Ref. [9] used the original operator arrangement $|S_i|^2$. Using this gives the term associated with $K_2$,

\[
\frac{1}{J^2} \sum_{M_i,M_f} \langle \Psi_{j,M_i}^{(F)} | \bar{K}_2(1,2) | \Psi_{j,M_f}^{(F)} \rangle = \sum_{lao'\gamma'} 2C_{a,l}^{a'} C_{a',l}^{a'} D_{a,a'} \hat{J}_1 \hat{J}_2 \sum_{KQ} (-1)^Q \left( \frac{1}{K^2} \right) [\text{direct} - \text{exchange}],
\]

where the direct and exchange terms are given by,

\[
direct = (-1)^{j_1 - j'_{j}} W(j_1j'_1j_2j'_2;KI) \left[ Z_{KQ}^{j_1\beta_1}(b) A_{K-Q}^{j_2\beta_2}(b) + A_{K-Q}^{j_1\beta_1}(b) Z_{K-Q}^{j_2\beta_2}(b) \right]
\]

\[
exchange = (-1)^{j_2 - j'_{j}} W(j_2j'_1j_1j'_2;KI) \left[ Z_{KQ}^{j_2\beta_2}(b) A_{K-Q}^{j_1\beta_1}(b) + A_{K-Q}^{j_2\beta_2}(b) Z_{K-Q}^{j_1\beta_1}(b) \right].
\]
For the projection-off term \( \bar{K}_3 \) we will require terms with the operator \((1 - \mathcal{S}_i)\), i.e.

\[
(\phi_{\mathcal{T}}^{m'} m | (1 - \mathcal{S}_i) | \phi_{\mathcal{T}}^m) = \int d\vec{r}_i \left( \phi_{\mathcal{T}}^{m'} m | (1 - \mathcal{S}_i) | \phi_{\mathcal{T}}^m \right)_{sp} \\
= \sum_{kj} (j' m' kj | jm) \int d\vec{r}_i (1 - \mathcal{S}_i) Z_{kj}(\vec{r}_i) \\
= \sum_{kj} (j' m' kj | jm) \mathcal{B}_{kj}^{\beta\beta'}(b),
\]

where we have defined the integral over the nucleon co-ordinates for projection-off part as,

\[
\mathcal{B}_{kj}^{\beta\beta'}(b) = \int d\vec{s}_i (1 - \mathcal{S}_i) \int dz_i Z_{kj}(\vec{r}_i) \\
(2.70)
\]

\( \mathcal{B}_{kj}^{\beta\beta'}(b) \) corresponds to the function \( \{ j' \ell' | \mathcal{I}_{KQ}(b) | j \ell \} \) of [9], though again, it was written in with the original operator arrangement \( \mathcal{S}_i \). Again, we are left to sum over the nucleon angular momentum projections \( m_1, m_2, m'_1 \) and \( m'_2 \),

\[
\sum_{m_1 m_2 m'_1 m'_2} (j_1 m_1 j_2 m_2 | I_{\mu} | j_1 m'_1 k_1 q_1 | j_1 m_1) \\
(\bar{K}_3 | k q | \bar{K}_3) \\
= \sum_{KQk'q'k''q''} (-1)^{l-1-j_1-j_2} (\mathcal{J}_2^{\beta\beta'} | k q | \mathcal{J}_2^{\beta\beta'}) \\
W(j'_2 k' K; k'' j_2) W(j_1 m_1 J; K I) \\
(2.72)
\]

Using this result we obtain

\[
\frac{1}{J_i^2} \sum_{M_i, M_f} \langle \Psi^{(F)}_{J_i M_i} | \bar{K}_3(1, 2) | \Psi^{(F)}_{J_f M_f} \rangle = \sum_{\mathcal{J}_1, \mathcal{J}_2} \sum_{D_{\alpha\alpha'}} \sum_{KQk'q'k''q''} \sum_{j''} \frac{(-1)^{l-1-j_1-j_2} + Q}{\hat{k}'} (k'' q'' K - Q | k' q') \\
\sum_{\mathcal{J}_1, \mathcal{J}_2} \text{[direct - exchange]},
\]

\( 2.73 \)
where \( \text{direct} \) and \( \text{exchange} \) are given as,

\[
\text{direct} = W(j_1 j_2' j_3; K I) 
\left[ (-1)^{j_1 - j_2' - j_3} W(j_1^* j_2' k K; k'' j_3) \bar{Z}^{j_1 j_2}_{K Q}(b) B^j_{K q'}(b) B^j_{j_3 q''}(b)^* + 
(-1)^{j_1 - j_2} j_2' W(j_1^* j_2' k' K; k'' j_1) B^j_{k' q'}(b) B^j_{j_2 q''}(b)^* \bar{Z}^{j_1 j_2}_{K Q}(b) \right],
\]

\[
\text{exchange} = W(j_2 j_1' j_3; K I) 
\left[ j_1' W(j_2 j_1' k' K; k'' j_1) \bar{Z}^{j_1 j_2}_{K Q}(b) B^j_{k' q'}(b) B^j_{j_3 q''}(b)^* + 
(-1)^{j_1 - j_1' + j_2 - j_3} j_1' W(j_2 j_1' k' K; k'' j_2) B^j_{k' q'}(b) B^j_{k q''}(b)^* \bar{Z}^{j_1 j_2}_{K Q}(b) \right].
\]

The diffractive-stripping cross section is obtained in a similar manner to the pure stripping cross section, that is, by integrating Eqs. 2.67 and 2.73 over the centre of mass impact parameter \( b \),

\[
\sigma_{\text{ds}}^{(j)} = \frac{1}{j_1^2} \sum_{M_i, M_f} \left\langle d\hat{b} \right| \mathcal{S}_c |^2 \left( \Psi^{(F)}_{J_i M_i} | \bar{K}_2(1,2) - \bar{K}_3(1,2) | \Psi^{(F)}_{J_f M_f} \right) \]

Typically it is estimated that the pure diffractive cross sections contributes \( \sim 8\% \) of the total cross section.

### 2.3 Fully correlated residue momentum distributions

We now consider the longitudinal momentum distribution of the heavy residue after the removal of two (typically) well-bound like-nucleons, as described in Refs. [19] and [20] (found in Appendices B.2 and B.3). The equivalent distribution in the case of one-nucleon
removal is strongly indicative of the orbital angular momentum of the removed nucleon and in the case of two-nucleon knockout the width of the residue momentum distribution has been shown, in approximate calculations, to be sensitive to the total angular momentum of the removed nucleons (see Refs. [29] and [17]). We consider only the longitudinal momentum, that in the beam direction, and not those perpendicular to this, which in single-nucleon knockout are susceptible to Coulomb effects and provide less information on the underlying projectile structure.

Essentially, measuring the momentum of the residue in the laboratory frame after the reaction probes the momentum of the nucleon pair in the projectile rest frame. The momentum components involved are shown in Fig. 2.5, where $\mathbf{K}_A$ and $\mathbf{K}_{A+2}$ are the projectile and residue momenta in the laboratory frame and $\mathbf{\kappa}_c$, $\mathbf{\kappa}_1$ and $\mathbf{\kappa}_2$ are the momenta of the core and two nucleons in the projectile rest frame. The projectile rest frame momenta sum to zero $\mathbf{\kappa}_1 + \mathbf{\kappa}_2 + \mathbf{\kappa}_c = 0$, and the incident projectile and reaction residue momenta are related by

$$\mathbf{K}_A = \left( \frac{A}{A+2} \right) \mathbf{K}_{A+2} + \mathbf{\kappa}_c.$$

(2.78)

The sum of the two-nucleon momenta is related to the incident projectile momentum and reaction reside momentum by

$$\mathbf{\kappa}_1 + \mathbf{\kappa}_2 = -\mathbf{\kappa}_c = \left( \frac{A}{A+2} \right) \mathbf{K}_{A+2} - \mathbf{K}_A.$$

(2.79)

The incident projectile typically has some narrow spread of momenta $\mathbf{K}_{A+2}$, which is typically measured in experiments. The residue momentum distribution thus reflects the momentum distribution of the two nucleons. A discussion of the motion of the individual nucleons is given in Ref. [17]. The narrowest components of the residue momentum distribution arise where the nucleon velocities are equal and opposite (counter-rotation) and when the plane of motion is perpendicular to the beam direction. The widest parts arise from components of the projectile wave function where the motion of both the nucleons is parallel or antiparallel to the beam direction.

Usually only the heavy residue is observed in experiment, but recent single-proton removal experiments on $^9$C and $^8$B have observed both the heavy residue and removed proton [30], and successfully used the summed energy of residue and nucleon to distinguish the stripping and diffractive mechanisms.
2.3. FULLY CORRELATED RESIDUE MOMENTUM DISTRIBUTIONS

Figure 2.5: Momentum components of the composite projectile. The core and valence nucleon projectile-rest-frame momenta are denoted by \( \vec{\kappa}_c \), \( \vec{\kappa}_1 \) and \( \vec{\kappa}_2 \). In the laboratory frame the momentum of the incident projectile is denoted by \( \vec{K}_{A+2} \) and the heavy residue by \( \vec{K}_A \). We consider only the momentum in the beam direction \( z \), referring to the components of the (vectorial) momenta in this direction by scalar versions of the above quantities, e.g. \( \kappa_c \).

2.3.1 Two-nucleon overlap momentum sampling

The two nucleon knockout reaction samples the momentum content of the projectile in a very particular way. One might envisage the target boring a cylindrical hole through the projectile, necessarily at or near the projectile surface so as not to disturb the projectile core. Approximate forms for the heavy residue momentum distribution [29, 17] have previously been obtained by sampling the momentum distribution for fixed \( s_1 \) and \( s_2 \) for a given angular separation \( \varphi_{12} \) of the two nucleons in the impact parameter plane, where it is assumed that the corresponding impact parameters are such that the nucleons are removed from the projectile. Note that in this approximate form, the reaction mechanism (stripping, diffraction) is not specified. Indeed, the \( S \)-matrices are neglected entirely, with the assumption being that given the co-ordinates \( s_1 \) and \( s_2 \) the nucleons will be removed with certainty. It is thus expected that the residue momentum distributions will be independent of the reaction mechanism (elastic, inelastic) if the volumes probed by the mechanisms are the same.

Following on from Section 2.2.4, we extend the concept of overlap sampling to momentum distributions. The two-nucleon position probability contains the integral over the core-nucleon \( z \)-co-ordinate and is the main source of two-nucleon correlations. We obtain the residue momentum distribution by using the completeness relation,

\[
\frac{1}{2\pi} \int d\kappa \exp[i(z - z')\kappa] = \delta(z - z'),
\]  

(2.80)
with $\kappa$ the momentum (wave number) of the nucleon. If integrated over the $dz'$ co-ordinate we obtain

$$\frac{1}{2\pi} \int dz' \int d\kappa \exp[i(z - z')\kappa] = 1.$$  \hspace{1cm} (2.81)

The lower and upper limits for both the $z'$ and $\kappa$ integrals are $-\infty$ and $+\infty$ and, this understood, these limits will be neglected for convenience. We insert the unit operator Eq. 2.81 into the spin-integrated two-body overlap wave function in the joint position probability $P_f(s_1, s_2)$ of Eq. 2.42 to give the joint position-momentum probability $\bar{P}_f(s_1, s_2, \kappa_c)$,

$$\bar{P}_f(s_1, s_2, \kappa_c) = \frac{1}{j_{j_1}^{M_j}} \sum_{M_f} \int d\kappa_1 \int d\kappa_2 \frac{\delta(\kappa_c + \kappa_1 + \kappa_2)}{(2\pi)^2} \times \left\langle \left| \int dz_1 \int dz_2 \exp(i\kappa_1 z_1) \exp(i\kappa_2 z_2) \Psi^{(F)}_{j_1 M_1} \right|^2 \right\rangle_{sp}. \hspace{1cm} (2.82)$$

Momentum conservation, $\kappa_c = -\kappa_1 - \kappa_2$, is ensured by the delta function. The final-state-exclusive differential cross section is then given by

$$\frac{d\sigma^{(f)}}{d\kappa_c} = \int d\vec{b} \int d\vec{s}_1 \int d\vec{s}_2 \bar{P}_f(s_1, s_2, \kappa_c) O_{ko}(c, 1, 2). \hspace{1cm} (2.83)$$

Again, it is $\bar{P}_f$ that contains the information on two-nucleon correlations and momentum, and the spatial sampling of this is determined by the reaction dynamics and $S$-matrices, described by $O_{ko}$. We reiterate that since the spatial sampling of the stripping (absorptive) and diffractive (elastic-breakup) mechanisms is expected to be similar, i.e. $O_{str}$ and $O_{ds}$ probe similar ranges of $s_i$, we would expect the heavy residue momentum distributions arising from the two mechanisms to be very similar on very general grounds.

### 2.3.2 Separation of integrals

The calculations of the residue momentum distributions involve an eight-dimensional integral that, for computational reasons, benefits from careful factorization of the integrals. We insert Eq. 2.81 into the single-nucleon overlap $(\phi_{\lambda}^{m'} | \phi_{\lambda}^{m})_{sp}$ and in doing so obtain two additional (four in total) integrals over the core-valence nucleon $z$ co-ordinate, and integrals over the valence nucleon wave numbers $\kappa_1$ and $\kappa_2$. We use the form of $(\phi_{\lambda}^{m'} | \phi_{\lambda}^{m})_{sp}$ as written with explicit (unsummed) orbital angular momenta projections $\lambda$, Eq. 2.51, in order to maintain symmetry between (c.f. Eq. 2.54) the integrals over $z$. The integrals
2.3. FULLY CORRELATED RESIDUE MOMENTUM DISTRIBUTIONS

over the core-nucleon co-ordinates for nucleon \( i \) are thus written as,

\[
I_i = \frac{1}{2\pi} \int d\vec{r}_i (1 - |S_i|^2) (\phi_j^m(\vec{r}_i)) \int dz_i \int d\kappa_i \exp[i(z_i - z_i')\kappa_i] |\phi_j^m|_n. \tag{2.84}
\]

Writing the nucleon co-ordinate \( \vec{r}_i \) integral in cylindrical co-ordinates,

\[d\vec{r}_i = ds \, s \, d\varphi \, dz,\]

gives \( I_i \) as

\[
I_i = \sum_{mm'\lambda'\sigma}' (\ell\lambda\sigma|jm) (\ell'\lambda'\sigma'|j'm') \int d\kappa_i \int ds_i \, s_i (1 - |S_i|^2) \exp[i(\lambda - \lambda')\varphi]
\times \left[ \frac{C_{\ell\lambda}}{\sqrt{2\pi}} \int dz_i \, u_{j\ell}(r_i) P_{\ell}^{j\lambda}(\cos \theta_i) \exp[i\kappa_i z_i] \right]
\times \left[ \frac{C_{\ell'\lambda'}}{\sqrt{2\pi}} \int dz'_i \, u_{j'\ell'}(r'_i) P_{\ell'}^{j'\lambda'}(\cos \theta'_i) \exp[-i\kappa_i z'_i] \right]
= \sum_{mm'\lambda'\sigma}' (\ell\lambda\sigma|jm) (\ell'\lambda'\sigma'|j'm') \int d\kappa_i \int ds_i \, s_i \mathcal{R}_{j\ell}(i) \mathcal{R}_{j'\ell'}(i)^* \mathcal{H}_{\lambda\lambda'}(i). \tag{2.85}
\]

Here the integrals over \( s \) are assumed to extend from 0 to +\( \infty \). We have defined the integral \( \mathcal{R}_{j\ell}(i) = \mathcal{R}_{\ell\lambda}(s_i, \kappa_i) \) over the \( z \)-component of the core-nucleon vector. This is essentially the one-dimensional (\( z \)-direction) Fourier transform of the single nucleon wave function,

\[
\mathcal{R}_{j\ell}(i) = \frac{C_{\ell\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dz_i \, u_{j\ell}(r_i) P_{\ell}^{j\lambda}(\cos \theta_i) \exp[i\kappa_i z_i]. \tag{2.86}
\]

The integral over the core-valence nucleon azimuthal \( \varphi_i \) is denoted \( \mathcal{H}_{\lambda\lambda'}(i) = \mathcal{H}_{\lambda\lambda'}(b, s_i) \), and is given by,

\[
\mathcal{H}_{\lambda\lambda'}(i) = \int_0^{2\pi} d\varphi_i (1 - |S_i|\sqrt{\vec{b} \cdot \vec{s}_i})^2 \exp[i\varphi_i(\lambda - \lambda')] \tag{2.87}
\]

The critical integrals over \( z_i \) and \( \varphi_i \) depend only on the projection of the orbital angular momentum, so the most efficient computational method would algebraically sum over all other projections. This fact also indicates the possible importance of the orbital angular momentum \( \ell \) over the nucleon total angular momentum \( j \) in determining the heavy residue momentum distribution and we will return to this in Chapter 4.

2.3.3 Angular momentum recoupling

The integrals \( \mathcal{R}_{\ell\lambda}(s_i, \kappa_i) \) and \( \mathcal{H}_{\lambda\lambda'}(b, s_i) \) depend on the projections of the orbital angular momenta only and so we sum over all other projections. We have a product of two integrals of the type \( I_i \) (Eq. 2.85), one for each nucleon. We begin with the two Clebsch-Gordan
coefficients of the form \((\ell \lambda s \sigma | j m)\) from each integral \(I_i\), and a further two Clebsch-Gordan coefficients arising from the two-nucleon overlap function. The starting point is then six Clebsch-Gordan coefficients summed over the projections \(m_1, m_2, m_1', m_2', \sigma_1, \sigma_2\) and \(\mu\),

\[
\mathcal{G}_{\lambda_1 \lambda_2 \lambda_1' \lambda_2'}^{\beta_1 \beta_2 \beta_1' \beta_2'} = \sum_{m_1 m_2 m_1' m_2' \sigma_1 \sigma_2 \mu} (j_1 m_1 j_2 m_2 | I \mu) (j_1' m_1' j_2' m_2' | I \mu) \\
(\ell_1 \lambda_1 s \sigma_1 | j_1 m_1) (\ell_1' \lambda_1' s \sigma_1 | j_1' m_1') \\
(\ell_2 \lambda_2 s \sigma_2 | j_2 m_2) (\ell_2' \lambda_2' s \sigma_2 | j_2' m_2'),
\]

(2.88)

where we have used the orthogonality of the spin-wave functions to give \(\chi_{\sigma \sigma} \chi_{\sigma' \sigma'} = \delta_{\sigma \sigma'}\). We first sum over the spin-projections \(\sigma\) using

\[
\sum_{\sigma} (\ell \lambda s \sigma | j m) (\ell' \lambda' s \sigma | j' m') = (-1)^{\ell_1 - s - m} \frac{\hat{j} \hat{j}'}{\ell \ell'} \sum_{k q} \hat{k} (\ell \lambda k q | \ell' \lambda') \\
(j - m \hat{j} \hat{j'} | k q) W(j \ell j' \ell'; sk).
\]

(2.89)

Having summed over both \(\sigma_1\) and \(\sigma_2\) using Eq. 2.89 we obtain

\[
\mathcal{G}_{\lambda_1 \lambda_2 \lambda_1' \lambda_2'}^{\beta_1 \beta_2 \beta_1' \beta_2'} = \sum_{k q} \sum_{m_1 m_1' m_2 m_2' \lambda_1 \lambda_2 \lambda_1' \lambda_2'} \frac{\hat{j}_1 \hat{j}_2 \hat{j}_1' \hat{j}_2'}{\ell_1 \ell_1'} (-1)^{\ell_1 + \ell_2 - 1 - m_2 - 2s} \\
(j_1 m_1 j_2 m_2 | I \mu) (j_1' m_1' j_2' m_2' | I \mu) \\
(j_1 - m_1 j_1' m_1' | k q) \\
(j_2 - m_2 j_2' m_2' | k' q') (\ell_1 \lambda_1 k q \ell_1' \lambda_1') (\ell_2 \lambda_2 k' q' \ell_2' \lambda_2') \\
W(j_1 \ell_1 j_1' \ell_1'; sk) W(j_2 \ell_2 j_2' \ell_2'; sk').
\]

(2.90)

We can now proceed to sum over the projections \(m_1, m_2, m_1', m_2'\) and \(\mu\),

\[
\sum_{m_1 m_2 m_1' m_2' \mu} (-1)^{-m_1 - m_2} (j_1 m_1 j_2 m_2 | I \mu) (j_1' m_1' j_2' m_2' | I \mu) \\
(j_1 - m_1 j_1' m_1' | k q) (j_2 - m_2 j_2' m_2' | k' q') \\
= \hat{\ell}^2 (-1)^{j_1 + j_2 + k + q} W(j_2 \ell_2 j_2' \ell_2'; sk_1 k_1 j_1 j_2) \delta_{k k_1} \delta_{q q_1},
\]

(2.91)

giving the final result,

\[
\mathcal{G}_{\lambda_1 \lambda_2 \lambda_1' \lambda_2'}^{\beta_1 \beta_2 \beta_1' \beta_2'} = (-1)^{\ell_1 + \ell_2 + j_1 + j_2 - 2s} \frac{\hat{j}_1 \hat{j}_2 \hat{j}_1' \hat{j}_2'}{\ell_1 \ell_1'} \hat{\ell}^2 \sum_{k q} (-1)^{k + q} \hat{k}^2 (\ell_1 \lambda_1 k q | \ell_1' \lambda_1') (\ell_2 \lambda_2 k - q | \ell_2' \lambda_2') \\
W(j_1 \ell_1 j_1' \ell_1'; sk) W(j_2 \ell_2 j_2' \ell_2'; sk) W(j_2 \ell_2 j_2' \ell_2') .
\]

(2.92)
In this final expression for $G_{\alpha_1 \alpha_2 \lambda_1 \lambda_2}^{\beta_1 \beta_2 \beta_1' \beta_2'}$ we have summed over the projection $\mu$. Later we will discuss final-state alignment effects (Section 6.2) for which we need to retain explicit dependence on $\mu$. At that point we revert to the original angular momentum coupling, Eq. 2.88.

Whilst the factor $G_{\alpha_1 \alpha_2 \lambda_1 \lambda_2}^{\beta_1 \beta_2 \beta_1' \beta_2'}$ certainly is convenient for practical computation of residue momentum distributions, how it affects their width as $I$ changes is not at all clear. The computation of the cross section requires a rather laborious sum over many combinations of $\lambda$ and $\alpha$, numbering from 10s to 10000s of combinations depending on the orbital angular momentum, number of configurations and $I$, all weighted by $G_{\alpha_1 \alpha_2 \lambda_1 \lambda_2}^{\beta_1 \beta_2 \beta_1' \beta_2'}$. We attempt to address this complication in Chapter 4.

### 2.3.4 Differential stripping cross section

Having recoupled the angular momenta we can write the differential cross section with respect to the heavy residue momentum. As with the momentum integrated cross section we have a coherent sum over the two-nucleon amplitudes for different nucleon configurations $\alpha$, but now have four integrals over the $z_i$ co-ordinate, and two over $\vec{s}_i$. Taking Eq. 2.49, and using Eqs. 2.86, 2.87 and 2.92, the heavy residue longitudinal momentum distribution can be written

\[ \frac{d\sigma_{str}^{(f)}}{d\kappa_c} = \int d\kappa_1 \int d\kappa_2 \delta(\kappa_1 + \kappa_2 + \kappa_c) \int d\vec{b} |S_c(b)|^2 \]

\[ \sum_{I_{ao'}} \frac{C_{\alpha'}^{J_{I} J_{I}} C_{\alpha}^{J_{I} J_{I}} D_{\alpha'} D_{\alpha}}{I^2} \]

\[ \sum_{\lambda_1 \lambda_2 \lambda'_1 \lambda'_2} \int ds_1 s_1 \int ds_2 s_2 \left[ \text{direct} - \text{exchange} \right], \tag{2.93} \]

where the direct and exchange terms are

\[ \text{direct} = G_{\alpha_1 \lambda_1 \lambda_2 \lambda'_1 \lambda'_2}^{\beta_1 \beta_2 \beta_1' \beta_2'} \left\{ H_{\lambda_1 \lambda_2'}(1) H_{\lambda_2 \lambda'_2}(2) R_{\lambda'_1 \lambda_1}(1) R_{\lambda_1 \lambda'_2}(2)^* \right\} \]

\[ + H_{\lambda_1 \lambda_2'}(2) H_{\lambda_2 \lambda'_2}(1) R_{\lambda'_1 \lambda_1}(2) R_{\lambda_1 \lambda'_2}(1)^* \right\}, \tag{2.94} \]

\[ \text{exchange} = (-)^{i_1 + j_2 - I} G_{\alpha_2 \lambda_1 \lambda_1' \lambda'_2}^{\beta_2 \beta_1' \beta_1 \beta_2'} \]

\[ \left\{ H_{\lambda_2 \lambda'_1}(1) H_{\lambda_1 \lambda_2'}(2) R_{\lambda'_2 \lambda_2}(1) R_{\lambda_2 \lambda'_1}(2)^* \right\} \]

\[ + H_{\lambda_2 \lambda'_1}(2) H_{\lambda_1 \lambda_2'}(1) R_{\lambda'_2 \lambda_2}(2) R_{\lambda_2 \lambda'_1}(1)^* \right\}. \tag{2.95} \]
The *direct* and *exchange* terms arise from the product of the anti-symmetrized two nucleon wave functions. The residue momentum distributions associated with the two terms are not necessarily the same, and only their sum is relevant to comparisons with data.

### 2.3.5 Differential stripping cross section in LS coupling

The (momentum-integrated) cross section was considered in LS-coupling in Ref. [8], where the key point was that the reaction is not selective on the total spin $S$. In the development of the longitudinal momentum distributions we noted that the only spin projection not algebraically summed over is that of the orbital angular momenta. This observation made, we consider the longitudinal momentum distributions in LS-coupling.

The advantage is that from the outset the only remaining projection is that of $\ell$, the rest being already summed. Since LS-coupling displays the symmetric and anti-symmetric parts of the two-nucleon overlap explicitly for a particular isospin $T$, we explicitly write isospin labels. We begin with the two nucleon overlap,

$$
\Psi^{(F)}_{J_i,M_i,T_i,\tau_i}(1,2) \equiv \langle \Phi^{(F)}_{J_f,M_f,T_f,T_f}(A)|\Psi_{J_i,M_i,T_i,\tau_i}(A,1,2) \rangle = \sum_{I_{\mu,T,\tau}} C^{I_{\mu,T,\tau}}_{J_fM_f,T_f,T_f} \langle I_{\mu,J_fM_f}|J_iM_i \rangle \langle T_{T_f}\tau_f|T_i\tau_i \rangle
$$

where $f$ and $F$ now incorporate the isospin labels $T_f$ and $\tau_f$ respectively in addition to the spin labels $J_f$ and $M_f$. In the present case of like-nucleon knockout we must have $T = |\tau| = 1$. The LS-coupled anti-symmetrized two nucleon wave function is [8]

$$
[\psi_{\ell_1j_1}(1) \otimes \psi_{\ell_2j_2}(2)]_{I_{\mu}}^{T_{\tau}} = D_{\alpha_1j_12} \sum_{\Lambda S \Sigma m_1 m_2} (\ell_1, \lambda_1, \ell_2, \lambda_2) (L \Lambda S \Sigma |I_{\mu}) \hat{L} \hat{S} \chi_{S \Sigma}(1,2) \chi_{T \tau}(1,2)
$$

where the nucleon-wave function $\psi_{\ell j}^{\lambda}(i) \equiv \psi_{\ell j}^{\lambda}(r_i)$ (c.f. $jj$-coupled version Eq. 2.25) is simply

$$
\psi_{\ell j}^{\lambda}(i) = u_{j\ell}(r_i) Y_{\ell \lambda}(\hat{r}_i).
$$

We follow a derivation similar to that in $jj$-coupling and begin from the LS-coupled
where, as before, summing over \( M_i \) and \( M_f \) gives an incoherent sum of \( I \) and \( \mu \),

\[
\sum_{M_i,M_f} \langle I\mu; J_f M_f | J_i M_i \rangle \langle I'\mu'; J_f M_f | J_i M_i \rangle = \delta_{\mu\mu'} \frac{\hat{J}_i^2}{I^2}. \tag{2.100}
\]

Inserting the \( LS \)-coupled anti-symmetrized two-nucleon wave function gives

\[
\frac{1}{J^2_i} \sum_{M_i,M_f} \langle \Psi^{(F)}_{J_i M_i T_i \tau_i} | \Psi^{(F)}_{J_f M_f T_f \tau_f} \rangle_{sp} = \sum_{I' \mu' \alpha' \alpha \alpha' \alpha} C_{\alpha \alpha'}^{J_i J_f IT_i T_f T'} C_{\alpha' \alpha}^{J_f J_f IT_i T_f T'} D_{\alpha \alpha'} \frac{\hat{J}_i^2 \hat{J}_f^2 \hat{L}_i \hat{L}_f \hat{S}_i \hat{S}_f}{I^2} \langle T_T \tau_T | T_i \tau_i \rangle \langle T'_T \tau'_T | T_i \tau_i \rangle \chi_{T_T \tau_T} \chi_{T_T} \\
\sum_{L \Lambda \Sigma \Lambda' \Sigma' \Lambda' \Sigma'} \chi_{L \Lambda \Sigma \Lambda' \Sigma'}^{\dag} (L' \Lambda' \Sigma' \Lambda) \langle \ell_1 \lambda_1 \ell_2 \lambda_2 | L \Lambda \rangle \langle \ell_1' \lambda_1' \ell_2' \lambda_2' | L' \Lambda' \rangle \\
(L \Lambda \Sigma \Lambda' \Sigma' | I\mu) \langle L' \Lambda' \Sigma' \Lambda | I\mu \rangle \chi_{L' \Lambda' \Sigma' L \Lambda} \chi_{L \Lambda} \chi_{L' \Lambda'} \chi_{L \Lambda} \chi_{L' \Lambda'} \chi_{L \Lambda} \chi_{L' \Lambda'} \\
\begin{bmatrix}
\ell_1 & s & j_1 \\
\ell_2 & s & j_2 \\
L & S & I
\end{bmatrix} \begin{bmatrix}
\ell_1' & s & j_1' \\
\ell_2' & s & j_2' \\
L' & S' & I
\end{bmatrix} \\
[\psi_{\ell_1 j_1}^{\lambda_1} (1) \psi_{\ell_2 j_2}^{\lambda_2} (2) - (-)^{S+T} \psi_{\ell_1 j_1}^{\lambda_1} (2) \psi_{\ell_2 j_2}^{\lambda_2} (1)] \\
[\psi_{\ell_1 j_1}^{\lambda_1} (1) \psi_{\ell_2 j_2}^{\lambda_2} (2) - (-)^{S'+T'} \psi_{\ell_1 j_1}^{\lambda_1} (2) \psi_{\ell_2 j_2}^{\lambda_2} (1)]^* \tag{2.101}
\]

Since the \( S \)-matrices are assumed to be spin-independent, the orthogonality of the spin and isospin wave functions gives an incoherent sum over \( S = 0 \) and \( S = 1 \) contributions,

\[
\chi_{S \Sigma' \Sigma}^\dag \chi_{S \Sigma} = \delta_{SS'} \delta_{\Sigma \Sigma'}, \tag{2.102}
\]

\[
\chi_{T_T \tau_T'}^\dag \chi_{T_T \tau_T} = \delta_{TT'} \delta_{\tau_T \tau_T'}. \tag{2.103}
\]
We can then proceed to sum over the projections of $I$ and $S$ ($\mu$ and $\Sigma$), using

$$\sum_{\mu \Sigma} (L\Lambda S\Sigma | I\mu) (L'\Lambda' S\Sigma | I\mu) = \delta_{LL'}\delta_{\Lambda\Lambda'} \frac{1}{L^2},$$

(2.104)

such that we now have an incoherent sum over both $L$ and $S$ combinations. The spin integrated two-nucleon overlap Eq. 2.101 is then,

$$\frac{1}{j_i^2} \sum_{M_iM_f} \langle \Psi^{(F)}_{J_i,n,M_iT_i\tau_i} | \Psi^{(F)}_{J,f,n,M_fT_f\tau_f} \rangle_{sp} = \sum_{I_0\alpha' T_1'} C_{\alpha }^{J_i,J_j,IT_iT_j} C_{\alpha'}^{J_i,J_j,IT_iT_j} D_{\alpha D_{\alpha'}} (T\tau T_fT_f\tau_f | T_iT_i)^2$$

$$\sum_{LS} \hat{j}_1 \hat{j}_2 \hat{j}_1' \hat{j}_2' S^2 \left\{ \begin{array}{ll} \ell_1 & s \ j_1 \\
 & & \\
 \ell_2 & s \ j_2 \\
 & & \\
 \ell_2' & s \ j_2' \end{array} \right\} \left\{ \begin{array}{ll} \ell_1' & s \ j_1' \\
 & & \\
 L & S \ I \\
 & & \\
 L & S \ I \end{array} \right\}$$

$$\sum_{\lambda_1\lambda_2\lambda_1'\lambda_2'} (\ell_1 \lambda_1 \ell_2 \lambda_2 | LA) (\ell_1' \lambda_1' \ell_2' \lambda_2' | LA)$$

$$\left[ \psi_{\ell_1 j_1}^{\lambda_1} (1) \psi_{\ell_2 j_2}^{\lambda_2} (2) - (-)^{S+T} \psi_{\ell_1 j_1}^{\lambda_1} (2) \psi_{\ell_2 j_2}^{\lambda_2} (1) \right]$$

$$\left[ \psi_{\ell_1 j_1}^{\lambda_1} (1) \psi_{\ell_2 j_2}^{\lambda_2} (2) - (-)^{S+T} \psi_{\ell_1 j_1}^{\lambda_1} (2) \psi_{\ell_2 j_2}^{\lambda_2} (1) \right]^*.$$

(2.105)

Structurally, Eq. 2.105 is very similar to Eq. 2.49. Following a parallel procedure as discussed previously for $jj$-coupling, we obtain the pure stripping cross section in $LS$-coupling, written, in terms of previously defined functions $R_i^J(i)$ and $\mathcal{H}_{\lambda\lambda'}(i)$ as

$$\frac{d\sigma_{str}^{(f)}}{dk_e} = \sum_{I_0\alpha' T_1'} C_{\alpha }^{J_i,J_j,IT_iT_j} C_{\alpha'}^{J_i,J_j,IT_iT_j} D_{\alpha D_{\alpha'}} (T\tau T_fT_f\tau_f | T_iT_i)^2$$

$$\sum_{LS} \hat{j}_1 \hat{j}_2 \hat{j}_1' \hat{j}_2' S^2 \left\{ \begin{array}{ll} \ell_1 & s \ j_1 \\
 & & \\
 \ell_2 & s \ j_2 \\
 & & \\
 \ell_2' & s \ j_2' \end{array} \right\} \left\{ \begin{array}{ll} \ell_1' & s \ j_1' \\
 & & \\
 L & S \ I \\
 & & \\
 L & S \ I \end{array} \right\}$$

$$\int d\kappa_1 \int d\kappa_2 \delta(\kappa_e + \kappa_1 + \kappa_2) \int d|b| S_e(b) |^2$$

$$\sum_{\lambda_1\lambda_2\lambda_1'\lambda_2'} (\ell_1 \lambda_1 \ell_2 \lambda_2 | LA) (\ell_1' \lambda_1' \ell_2' \lambda_2' | LA) [direct - exchange].$$

(2.106)
where the direct and exchange terms are

\[
direct = \left\{ \mathcal{H}_{\lambda_1 \lambda_1'} (1) \mathcal{H}_{\lambda_2 \lambda_2'} (2) \mathcal{R}_{\ell_1 \lambda_1'} (1) \mathcal{R}_{\ell_2 \lambda_2'} (2)^* \right. \\
+ \left. \mathcal{H}_{\lambda_1 \lambda_1'} (2) \mathcal{H}_{\lambda_2 \lambda_2'} (1) \mathcal{R}_{\ell_1 \lambda_1'} (2) \mathcal{R}_{\ell_2 \lambda_2'} (1)^* \right\},
\]

\[
exchange = (-1)^{S+T} \left\{ \mathcal{H}_{\lambda_2 \lambda_1'} (1) \mathcal{H}_{\lambda_1 \lambda_2'} (2) \mathcal{R}_{\ell_2 \lambda_2'} (1) \mathcal{R}_{\ell_1 \lambda_1'} (2)^* \right. \\
+ \left. \mathcal{H}_{\lambda_2 \lambda_1'} (2) \mathcal{H}_{\lambda_1 \lambda_2'} (1) \mathcal{R}_{\ell_2 \lambda_2'} (2) \mathcal{R}_{\ell_1 \lambda_1'} (1)^* \right\}.
\]

Unlike the $jj$-coupled version, no re-coupling of the angular momenta is required to compact the angular momentum algebra. The above equation is very similar to the previous version, Eq. 2.93, but with the opaque $g^{\beta_1 \beta_2 \beta_1' \beta_2'}_{\lambda_1 \lambda_2 \lambda_1' \lambda_2'}$ replaced with two $9j$ and two Clebsch-Gordan coefficients, the physical meaning of which is transparent. We note that the summation of the direct and exchange terms over $\lambda$ and $\Lambda$ is independent of $I$ for a particular configuration $\alpha$. The significance of $I$ is that it determines the relative weights of different $L$ via the two $9j$ symbols. Fundamentally it is $L$ that determines the residue momentum distribution shape, with $I$ being of secondary importance. We return to this point in Chapter 4. In order to facilitate later discussions, we define the residue momentum distribution for a particular pair of configurations $(\alpha,\alpha')$ with total orbital angular momentum $L$, by taking the last two lines of Eq. 2.106,

\[
\frac{d\sigma^{\alpha'\alpha'^L}}{d\kappa_c} = \int d\kappa_1 \int d\kappa_2 \delta(\kappa_c + \kappa_1 + \kappa_2) \int d\vec{b} \left| S_c(b) \right|^2 \\
\sum_{\lambda_1 \lambda_2 \lambda_1' \lambda_2'} (\ell_1 \lambda_1 \ell_2 \lambda_2 | \Lambda \Lambda) (\ell_1' \lambda_1' \ell_2' \lambda_2' | \Lambda \Lambda) \left[ \text{direct} - \text{exchange} \right].
\]

### 2.3.6 Diffractive-stripping momentum distributions

Thus far we have considered only the residue momentum distribution resulting from pure stripping events. The situation is rather more complicated for the diffractive-stripping contributions, as illustrated in Section 2.2.6 for the (momentum integrated) diffractive-stripping cross section. The diffractive-stripping operator $\mathcal{K}_{ds} = K_2 - K_3$, contains two parts, the second of which insures the elastically scattered nucleon does not remain bound to the core. The first term of the diffractive-stripping operator $K_2$ is similar in structure to the pure stripping term and the corresponding momentum distribution is similarly calculated. However, it is not immediately clear how to calculate the momentum distribution arising from the second (projection-off) term $K_3$ due to the sum over bound states. We might attempt to approximate the stripping-diffraction momentum distribution by con-
2.4. **UNCORRELATED TWO-NUCLEON KNOCKOUT**

Considering only the first term $\mathcal{K}_2$, but, as was discussed, the form of the $S$-matrix implies that, if using $\mathcal{K}_{ds}$ as was originally written, the largest contributions from $\mathcal{K}_2$ arise when the diffracted nucleon is at the largest impact parameters and there is essentially no interaction with the target. This necessitates a very large cancellation between the terms $\mathcal{K}_2$ and $\mathcal{K}_3$, such that we would not expect the momentum distribution arising from $\mathcal{K}_2$ to be representative of the total diffractive-stripping distribution of $\mathcal{K}_{ds}$.

Indeed, since the nucleon absorption probability is one minus the probability for elastic scattering ($1 - |S_i|^2$), the operator as originally written suggests that the nucleon will be diffracted in a spatial region that is not characterized by absorption, i.e. the two nucleons will not be spatially localized, and we would expect the diffraction events to occur when the diffracted nucleon is not near the target-side surface of the projectile. This considered, we would expect $\mathcal{K}_2$ to give a rather poor description of the heavy residue momentum distribution, in shape as well as magnitude.

Making the substitution $S_i \rightarrow (1 - S_i)$, as discussed in Section 2.2.6, instead emphasises the surface localisation of the cross section, reducing the cancellation of the projection-off term $\overline{\mathcal{K}}_3$. The significance of $\overline{\mathcal{K}}_3$ will be shown to be small and the distribution arising from the first term $\overline{\mathcal{K}}_2$ can be calculated in a similar way to that from pure stripping. The only required change is an adjustment of the azimuthal integral for the diffracted nucleon in Eqs. 2.94 and 2.95,

$$\mathcal{H}_{\lambda\lambda'}(i) \rightarrow \overline{\mathcal{H}}_{\lambda\lambda'}(i) = \int_0^{2\pi} d\varphi \left[ 1 - S_i(b_i) \right]^2 \exp[i\varphi (\lambda - \lambda')],$$  

(2.110)

which corresponds to $\overline{\mathcal{K}}_2$, and to include additional terms to account for the diffraction of both nucleons 1 and 2. A comparison of the stripping and diffractive-stripping momentum distribution is presented in Section 6.4, for both the rewritten operator $\overline{\mathcal{K}}_2$ and $\mathcal{K}_2$, showing the distributions calculated on the basis of $\overline{\mathcal{K}}_2$ to be essentially identical to those of pure stripping and to accurately represent the total diffractive-stripping cross section.

### 2.4 Uncorrelated two-nucleon knockout

Early calculations for two-nucleon knockout [7] used an uncorrelated model, treating the two nucleons as entirely independent, save for being bound to the same centre. In particular no angular momentum coupling of the nucleons was considered and antisymmetrization of the two nucleons was neglected. In parallel with single-nucleon knockout, where the cross section factorizes into a single-particle cross section and spectroscopic factor, a *two-particle* cross section is obtained, giving a base measure of the cross section. One may then extract a *two-particle spectroscopic factor* from experiment, which is comparable to
simple estimates via coefficients of fractional parentage. Whilst the structural description is simplistic, such estimates set the expected magnitude of the cross section.

Here we consider only events where both nucleons are stripped from the projectile. The reaction dynamical (two-particle cross section) and structural (two-particle spectroscopic factor) are separated, as in single-nucleon knockout. The final state appears only in the two-particle spectroscopic factor, so the two-particle cross section is independent of the final state. In accordance with this simple structural picture, we consider only a single nucleon pair configuration. Again, we make the no-recoil approximation, such that the core impact parameter is coincident with that of the centre of mass and the core $S$-matrix can then be removed from the integrals over $\vec{r}_i$. The uncorrelated knockout (stripping) cross section \[7\] is given by

$$
\sigma_{\text{str}}^{\text{unc}} = \int d\vec{b} |S_c|^2 \prod_{i=1,2} \left[ \frac{1}{\ell_i^2} \sum_{m_i} \langle \phi_{j_i}^{m_i} | (1 - |S_i|^2) | \phi_{j_i}^{m_i} \rangle \right],
$$

where $\phi_{j_i}^{m_i}$ is the single-nucleon wave function Eq. 2.25 and $\langle ... \rangle$ indicates integration over (single-nucleon) position and spin co-ordinates.

Previous approximate calculations \[7\] for the heavy residue momentum distribution used the convolution of two single-nucleon knockout residue momentum distributions, reasonable if the nucleons are localised near to the projectile surface. The agreement of this uncorrelated calculation and the final-state inclusive residue momentum distribution was rather good in the case of $^{28}\text{Mg}(\sim 2p)$ where essentially the full two-proton removal strength is to bound states of $^{26}\text{Ne}$. The exact uncorrelated residue longitudinal momentum distribution, written in terms of the functions defined in Section 2.3.2, is actually

$$
\frac{d\sigma_{\text{str}}^{\text{unc}}}{d\kappa_c} = \int d\kappa_1 \int d\kappa_2 \delta(\kappa_c + \kappa_1 + \kappa_2) \int d\vec{b} |S_c(b)|^2
\prod_{i=1,2} \left[ \frac{1}{\ell_i^2} \sum_{\lambda_i} \int ds_i \mathcal{H}_{\lambda_i}(i) |R_{i,\lambda_i}^b(i)|^2 \right].
$$

The primary difference between this and the fully-correlated calculations is the absence of the final state spin and angular momentum coupling. An immediate consequence of this is that the residue momentum distributions are independent of the final state populated, and depend only on the orbital angular momenta of the assumed nucleon pair configuration. This contradicts the result of preliminary fully-correlated results \[16, 17\]. The one case where the uncorrelated limit is in agreement with the full calculations is the case of $[s_{1/2}]^2$ (pure $s$-wave) removal. Here the only contributions in both correlated and uncorrelated cases must have $\lambda = 0$. Results from uncorrelated calculations will be discussed further in Section 6.1.
Since it was shown that the uncorrelated model reasonably describes the final-state inclusive residue momentum distribution in the case of $^{28}\text{Mg}(\sim 2p)$, it would be interesting to analytically sum the fully-correlated calculations over the final states for theoretical comparison to the uncorrelated limit shown above. This will not be pursued further here.
Chapter 3

Practical details

Here we discuss some of the practical details for the calculations that follow. We briefly
discuss the shell model codes used in calculating spectroscopic factors, two-nucleon am-
plitudes and excited state energy levels, the exact details of which will be given where
they are relevant. All calculations require a valence nucleon radial wave functions and
the form of the potential used to bind the nucleon are also discussed here, with details
of the potential parameters discussed in later Sections. In order to compare theoretical
projectile-rest frame residue momentum distributions with experiment we must consider
experimental broadening effects which are also discussed.

3.1 Single-nucleon wave functions

The essential inputs in calculating the required single particle wave function is the (exper-
imental) nucleon separation energy and the geometry of the binding potential. In most
cases the nucleon separation energies are well known and can be taken from the literature
[48], but for the most exotic nuclei they may be estimated from systematics. The nucleon
radial wave functions are calculated using a Woods-Saxon (central plus spin-orbit) [49]
plus Coulomb potential, with a fixed spin-orbit potential depth $V_{so}$ [50, 42] using the same
geometry parameters as the nuclear part. The potential is then

$$U(r) = V(r) + V_{so}(r)(\vec{\ell} \cdot \vec{s}) + V_C(r)$$  \hspace{1cm} (3.1)
where the nuclear $V(r)$, spin-orbit $V_{so}$ and Coulomb $V_C$ parts are given by

\begin{align}
V(r) &= -\frac{V_0}{1 + \exp[(r - R)/a]}, \\
V_{so}(r) &= -\frac{2V_0^{so}}{ra} \exp[(r - R)/a]\left[1 + \exp[(r - R)/a]\right]^2, \\
V_C(r) &= \frac{Z_c Z_v e^2}{r}, \quad r > R_C, \\
&= \frac{Z_c Z_v e^2}{2R_C} \left[3 - \frac{r^2}{R_C^2}\right], \quad r \leq R_C.
\end{align}

$V_0$ is the Woods-Saxon potential depth, $R$ the potential radius often expressed as $R = r_0 A^{1/3}$ and $a$ the diffuseness parameter which determines the sharpness of the potential surface, with larger values of $a$ giving a softer surface. $Z_c$ and $Z_v$ are the proton numbers of the core and valence particle respectively.

The geometry is constrained by taking the root mean square radius and binding energy for a given orbital from a Hartree-Fock calculation and fitting the radius parameter $r_0$ of the Woods-Saxon well so as to reproduce the Hartree-Fock r.m.s. radius and separation energy for a particular potential diffuseness $a$ and spin-orbit depth $V_{so}$ [4]. In the case of neutron knockout from neutron rich carbon isotopes, presented in Appendix B.1, a standard geometry potential has been used in order to study the systematics of cross section across the isotopic chain.

With this fitted potential geometry, the nucleon wave function is calculated at the experimental effective separation energy $B_N^{eff}$, which for one- and two-nucleon knockout are,

\begin{align}
B_N^{eff} &= S_N + E_f, \\
B_N^{eff} &= \frac{S_{2N} + E_f}{2},
\end{align}

respectively, where $E_f$ is the excitation of the residue final state above the ground state. $S_N$ is the ground state to ground state single-nucleon separation energy and $S_{2N}$ the two-nucleon separation energy. In most cases the excitation of the final states is known, either from in-beam $\gamma$-ray spectroscopy or from the literature, but where this information is not available shell-model energies have been used.

Single-nucleon removal cross sections have been shown to be relatively insensitive to the geometry of the potential used. In Ref. [4] it was shown that the essential parameter is the root-mean-square radius of the radial wave function. The use of the Hartree-Fock sizes and separation energies allows us to realistically constrain the overall size of the radial wave function.
3.2 Shell-model calculations

Calculation of knockout cross sections requires a spectroscopic factor (single-nucleon knockout, see Appendix B.1) or two-nucleon amplitudes (two-nucleon knockout) that describe the parentage of a particular final state in the projectile ground state. In cases where final-state exclusive cross sections are not measured, final state excitation energies and spin-parity assignments are also required. This information is taken from shell model calculations performed using the codes oxbash [43] and NuShellX@MSU [51, 52]. The details of the model spaces and interactions used can be found in each of the following subsections. The model space describes the active orbitals available for a calculation, which may encompass a single, e.g. sd or multiple major shells e.g. \( p - sd - pf \). In some cases the model spaces are restricted such that only certain partitions (combinations of nucleons in orbitals) are allowed, or such that only a certain number of nucleons \( n \) can be excited across a major-shell closure, referred to as \( n\hbar \omega \) excitations. The interactions chosen have either been used previously or have been recommended by the authors of the codes. The calculations are either performed in an isospin basis, where proton and neutron orbitals are equivalent, or in a \( pn \) basis where the proton and neutron orbitals are treated separately. Two-nucleon amplitudes from the former must be multiplied by the appropriate isospin Clebsch-Gordan coefficient (see e.g. Section 2.3.5).

When considering residue momentum distributions the angular momentum content of the overlaps is important i.e. the particular nucleon quantum numbers and total angular momentum coupling, which are the primary factors determining the residue momentum shape. The absolute values of spectroscopic factors and two-nucleon amplitudes set the absolute scale of the cross section. The relative strengths and phases of different two-nucleon configurations are important for determining the final-state exclusive momentum distribution shape (though these effects are generally expected to be subtle; see Section 6.6). The shape of final-state inclusive momentum distributions will depend on the relative strengths of different final states and offers a check on (momentum-integrated) cross section measurements.

3.3 Experimental broadening of heavy residue momentum distributions

In order to compare to experimental residue momentum distributions, we must convolute the theoretical distributions calculated in the projectile rest frame with the experimental resolutions. We must first apply a Lorentz stretch due to the relativistic velocity of the residue. The Lorentz factor can be calculated from the definition of the relativistic energy,
the sum of the laboratory kinetic energy \( E_K \) and rest mass \( m_0 \), rearranged to give

\[
\gamma = \frac{E_K + m_0 c^2}{m_0 c^2}.
\]

(3.8)

As the beam passes through the reaction target, it loses energy due to (Coulomb) interactions with atoms in the target [53]. The rate of energy loss is related to the charge of the nucleus in question, so in reactions where the incident projectile and reaction residue have different charge (e.g. proton knockout), the rate of energy loss will be different before and after the reaction. Depending on where the reaction vertex is located within the target, the differing rates of energy loss cause a broadening of the residue momentum distribution. Since two-neutron removal reactions do not alter the charge in the reaction, this problem is negligible in these cases. Such broadening can be reduced by using thinner targets so that the reaction vertex is better constrained, but the price to paid is a reduced experimental yield.

In addition to broadening induced by material in the path of the beam, the incident secondary projectile beam is itself not mono-energetic and has some momentum spread. This is estimated either from the acceptance set-up of the spectrometer or by measuring the unreacted beam after the reaction target. Adjustments for experimental broadening (target broadening, incident beam profile) require convolution with the broadening function \( B(K_A) \) and the theoretical, Lorentz boosted, residue momentum distribution \( \sigma(\kappa) \),

\[
(\sigma \otimes B)(K_A) = \int_{-\infty}^{+\infty} \sigma(\kappa) \ B(|K_A - \kappa|) \ d\kappa.
\]

(3.9)

The broadening function can also be estimated using the spectrometer simulator LISE++ [54]. Alternatively the differential energy loss in the target may be approximated by a square distribution and the beam profile as a Gaussian, necessitating several convolutions. The convolution routines written were verified against the convolution of two Gaussian functions, which can be calculated analytically.

### 3.4 Details of computer codes

A number of codes were used in the calculations presented here, many of which have been used extensively in previous (published) work on knockout reactions. The folding-model \( S \)-matrices were calculated using the codes \texttt{FRONT\_HF} and \texttt{SMATHF} and the nucleon radial wave functions were calculated using the code \texttt{NBND} [55]. Projectile core density profiles calculated using the spherical Hartree-Fock program \texttt{DENS}, which is part of the \texttt{OXBASH}
Residue momentum distributions following two-nucleon knockout were calculated using the code TWONK3 [56], written in FORTRAN95, with some subroutines using FORTRAN77. The code is structured for computational efficiency. Many integrals are calculated and stored in memory, which results in large memory requirements, particularly for large orbital angular momenta, many two nucleon configurations or fine numerical integration steps. The integrals are calculated by a mixture of Simpsons Rule and Gaussian quadrature methods, the subroutines for which were provided by J. A. Tostevin. Associated Legendre polynomials were determined using explicit values for $\ell = 0, 1$ and 2, and the recursion relations given by Ref. [57]. The momentum-integrated cross sections were verified with an optimised version of the code TWONGEN3 [55].

Calculations of momentum-integrated single-nucleon removal cross sections used the program NTRASH [55]. A derivative of the TWONK3 program, ONEKO, was used to calculate the single-nucleon removal momentum distributions shown in Appendix B.1. Calculations for Hankel radial wave functions used subroutines taken from Ref. [58]. Modified spherical Bessel functions were calculated using the code of Ref. [59].
Chapter 4

Angular correlations

The primary expectation from early, approximate estimates for residue momentum distribution following two-nucleon removal was that the width of the distribution depends strongly on the total angular momentum of the removed nucleons $I$ [29, 17]. Having derived the full formalism, it is not transparent from Eq. 2.93 why the residue momentum distribution should depend strongly on the total angular momentum $I$, or, indeed, any other factors. Whilst it is clear that the $z$-directional Fourier transform of the single-nucleon wave function $R_{j\ell\lambda}$ must have some dependence on $\lambda$, each value of $I$ will sample all $\lambda$ in a complicated manner with weight given by the non-trivial angular momentum function $G_{\lambda_1\lambda_2\lambda'_1\lambda'_2}^{\beta_1\beta_2\beta'_1\beta'_2}$ (see Eq. 2.92). At this stage, even the dependence of $R_{j\ell\lambda}$ on $\kappa$ or $s$ is not clear, let alone how it might change as $\lambda$ and $\ell$ vary. In order to investigate the two nucleon correlations further, we follow two strategies; i) simplify the angular momentum algebra to remove the extensive sums over $\lambda$, and ii) simplify $R_{j\ell\lambda}$ using some approximation for $u_{\ell j}(r)$ to make the dependence on $\lambda$ more transparent. The latter we discuss in Chapter 5 and the former we consider presently.

The angular separation $\varphi_{12}$ of the two nucleons in the impact parameter ($xy$) plane has previously been discussed in Refs. [16] and [17]. It was written in terms of $P_f(s_1, s_2, \varphi_{12})$ (see Eq. 2.42), the two-nucleon position probability for a particular final state $f$. This incorporates the integrals over the $z$-components of the nucleon position vectors $\vec{r}$, and shows a strong variation with the total angular momentum $I$. Here we take the angular dependent parts of the two-nucleon overlaps and, by summing over all projections, show how the strong dependence of $P_f$ on the total angular momentum $I$ arises naturally from the angular momenta and angular dependence of the spherical harmonics. In doing so, we show that differences in the residue momentum distribution for a particular pair coupled to $I$ are the result of the angular correlation of the two nucleons in the projectile ground state. We then express the same angular correlation function in LS coupling to show that, in fact, the angular correlation of the two nucleons is a consequence of the total
orbital angular momentum $L$, and that $I$ is of secondary importance. In either case we have a strong sensitivity to the final state populated.

### 4.1 Angular correlation function in $jj$-coupling

The heavy residue momentum distribution depends explicitly on the projections of the orbital angular momenta $\ell$, $\lambda$, and previously all other angular momentum projections have been summed over. Here we do not explicitly consider momentum distributions, but wish to investigate the influence of the total angular momentum $I$ on two-nucleon correlations. We take all projection dependent parts of the previous starting point, Eqs. 2.49 and 2.50, namely the four spherical harmonics and accompanying Clebsch-Gordan coefficients embedded in the single nucleon wave functions $\phi_j^m(i)$, and the further two Clebsch-Gordan coefficients from the anti-symmetrized two-nucleon wave function. We consider only the first direct term of Eq. 2.49 and so begin with

$$
\Gamma^I_{jj}(\hat{r}_1, \hat{r}_2) = \frac{1}{I^2} \sum_{m_1 m_2 m'_1 m'_2 \lambda_1 \lambda_2 \lambda'_1 \lambda'_2 \sigma_1 \sigma_2} \langle j_1 m_1 j_2 m_2 | I \mu \rangle \langle j'_1 m'_1 j'_2 m'_2 | I \mu \rangle \langle \ell_1 \lambda_1 \sigma_1 | j_1 m_1 \rangle \langle \ell_2 \lambda_2 \sigma_2 | j_2 m_2 \rangle \langle \ell'_1 \lambda'_1 \sigma_1 | j'_1 m'_1 \rangle \langle \ell'_2 \lambda'_2 \sigma_2 | j'_2 m'_2 \rangle Y_{\ell_1 \lambda_1}(\hat{r}_1) Y_{\ell_2 \lambda_2}(\hat{r}_2) Y_{\ell'_1 \lambda'_1}(\hat{r}_1)^* Y_{\ell'_2 \lambda'_2}(\hat{r}_2)^*. 
$$

(4.1)

All the dependence on $I$ is encapsulated in the Eq. 4.1, so that for a particular pair of configurations (combination of orbital pairs $\alpha$ and $\alpha'$ i.e. set nucleon quantum numbers $n_1 \ell_1 j_1, n_2 \ell_2 j_2, n'_1 \ell'_1 j'_1$ and $n'_2 \ell'_2 j'_2$), differences in the residue momentum distribution for different $I$ must be a result of differences arising from the above summation. The two-nucleon amplitudes also depend on the total angular momentum $I$, but act only as factors when combining combinations (of the same $I$) from different configurations. The above is written for the first direct term only. The second direct term and exchange terms may be obtained by appropriate permutation of the angular momenta labels and inclusion of an additional phase factor.

Using the properties of the spherical harmonics [46], the pairs of spherical harmonics...
with the same argument $\hat{r}_i$ can be combined to obtain

$$
\Gamma^{I}_{jj}(\hat{r}_1, \hat{r}_2) = \frac{1}{I^2} \sum_{m_1,m_2,m'_1,m'_2,\lambda_1\lambda_2\lambda'_1\lambda'_2,\sigma_1\sigma_2\mu} (j_1 m_1 j_2 m_2 | I\mu)(j'_1 m'_1 j'_2 m'_2 | I\mu)(j_1 m_1 j_2 m_2 | I\mu)
$$

\begin{align*}
& (j'_1 m'_1 j'_2 m'_2 | I\mu) (\ell_1 \lambda_1 s \sigma_1 | j_1 m_1) (\ell_2 \lambda_2 s \sigma_2 | j_2 m_2) \\
& (\ell'_1 \lambda'_1 s \sigma_1 | j'_1 m'_1) (\ell'_2 \lambda'_2 s \sigma_2 | j'_2 m'_2) (-1)^{\lambda'_1 + \lambda'_2} \\
& \sum_{k_1 q_1} \frac{\hat{\ell}_1 \hat{\ell}'_1}{\sqrt{4\pi k_1}} (\ell_1 \lambda_1 \ell'_1 - \lambda'_1 | k_1 q_1) (\ell_1 0 \ell'_1 0 | k_1 0) Y_{k_1 q_1}(\hat{r}_1) \\
& \sum_{k_2 q_2} \frac{\hat{\ell}_2 \hat{\ell}'_2}{\sqrt{4\pi k_2}} (\ell_2 \lambda_2 \ell'_2 - \lambda'_2 | k_2 q_2) (\ell_2 0 \ell'_2 0 | k_2 0) Y_{k_2 q_2}(\hat{r}_2). \\
& (4.2)
\end{align*}

In doing so we have introduced sums over intermediate angular momenta $k_1$ and $k_2$, and the relevant projections $q_1$ and $q_2$. The values of $k_i$ are restricted by the two parity Clebsch-Gordans $|\ell 0 \ell' 0 k 0\rangle$, such that $\ell_i + \ell'_i + k_i$ must be even.

We now evaluate the sums over the projections of $\ell_i$ and $\ell'_i$ ($\lambda_i$ and $\lambda'_i$) and the spin $s$ ($\sigma_i$) for each nucleon, by twice using

\begin{align*}
\sum_{\lambda,\lambda'_i,\sigma_i} (-1)^{\lambda'_i} (\ell_i \lambda_i s \sigma_i | j_i m_i) (\ell'_i \lambda'_i s \sigma_i | j'_i m'_i) (\ell_i \lambda_i \ell'_i - \lambda'_i | k_i q_i) \\
= (-1)^{m'_i} (-1)^{\ell_i + 2s - j'_i - k_i} W(j_i \lambda_i, \ell_i \ell'_i | j_i m'_i, k_i q_i).
\end{align*}

(4.3)

to give the result,

\begin{align*}
\Gamma^{I}_{jj}(\hat{r}_1, \hat{r}_2) = \frac{1}{I^2} \sum_{k_1 k_2 q_1 q_2} \frac{\hat{\ell}_1 \hat{\ell}'_1 \hat{\ell}_2 \hat{\ell}'_2 \hat{j}_1 \hat{j}'_1 \hat{j}_2 \hat{j}'_2}{4\pi k_1 k_2} (-1)^{\ell_1 + \ell_2 - j'_1 - j'_2 - k_1 - k_2} \\
(\ell_1 0 \ell'_1 0 | k_1 0) (\ell_2 0 \ell'_2 0 | k_2 0) Y_{k_1 q_1}(\hat{r}_1) Y_{k_2 q_2}(\hat{r}_2) \\
W(j_1 \lambda_1, \ell_1 \ell_1 | j_1 m_1, k_1 q_1) W(j_2 \lambda_2, \ell_2 \ell_2 | j_2 m_2, k_2 q_2) \\
\sum_{m_1,m_2,m'_1,m'_2,\mu} (-1)^{m'_1 + m'_2} (j_1 m_1 j_2 m_2 | I\mu)(j'_1 m'_1 j'_2 m'_2 | I\mu) \\
(j_1 - m_1 m'_1 | k_1 q_1) (j_2 - m_2 m'_2 | k_2 q_2).
\end{align*}

(4.4)
We are now able to evaluate the sums over \( m_1, m_2, m'_1, m'_2 \) and \( \mu \) using,

\[
\sum_{m_1m_2m'_1m'_2\mu} (-1)^{m_1+m'_1} (J_1m_1j_2m_2|I\mu) (J'_1m'_1j'_2m'_2|I\mu)
\]

\[
(j_1 - m_1j'_1m'_1|k_1 - q_1) (j_2 - m_2j'_2m'_2|k_2 - q_2)
\]

\[
= \hat{T}^2(-1)^{n+1}(-1)^{2j_1+j'_1-j'_2-k}W(j_1Ij_2;j'_2j_1')\delta(k_1, k_2)\delta(q_1,-q_2).
\]  

We now substitute \( k_1 = k_2 = k \) and \( q_1 = -q_2 = q \) to give

\[
\Gamma^I_{jj}(\hat{r}_1, \hat{r}_2) = \frac{\ell_1\ell'_1\ell_2\ell'_2j_1j_2j'_1j'_2}{4\pi} (-1)^{\ell_1+\ell_2} \sum_k (-1)^{\ell-k} \frac{1}{k^2} (\ell_10\ell_200|k0) (\ell_20\ell'_200|k0)
\]

\[
W(j_1skl'; \ell_1j'_1) W(j_2skl'; \ell_2j'_2) W(j_1Ikj'_2; j_2j'_1) Y_{kq}(\hat{r}_1) Y_{k-q}(\hat{r}_2).
\]  

Finally we combine the spherical harmonics using the addition theorem [46],

\[
\sum_q (-1)^q Y_{kq}(\hat{r}_1)Y_{k-q}(\hat{r}_2) = \frac{k^2}{4\pi} P_k(\cos \omega),
\]

where \( \omega \) is the angular separation of \( \hat{r}_1 \equiv (\theta_1, \varphi_1) \) and \( \hat{r}_2 \equiv (\theta_2, \varphi_2) \) given by

\[
\cos \omega = \hat{r}_1 \cdot \hat{r}_2 = \sin \theta_1 \sin \theta_2 \cos(\varphi_1 - \varphi_2) + \cos \theta_1 \cos \theta_2.
\]  

This gives the angular correlation function \( \Gamma^I_{jj}(\omega) \equiv \Gamma^I_{jj}(\hat{r}_1, \hat{r}_2) \),

\[
\Gamma^I_{jj}(\omega) = \frac{\ell_1\ell'_1\ell_2\ell'_2j_1j_2j'_1j'_2}{(4\pi)^2} (-1)^{\ell_1+\ell_2} \sum_k (-1)^{\ell-k} (\ell_10\ell_200|k0) (\ell_20\ell'_200|k0)
\]

\[
W(j_1skl'; \ell_1j'_1) W(j_2skl'; \ell_2j'_2) W(j_1Ikj'_2; j_2j'_1) P_k(\cos \omega).
\]

Having begun with a function depending on the directions \( \hat{r}_1 \) and \( \hat{r}_2 \), we are left with a function that depends only on the angular difference \( \omega \) of the two nucleon direction vectors \( \hat{r}_1 \) and \( \hat{r}_2 \). Previous calculations for \( P_I(s_1, s_2, \varphi_{12}) \) (Eq. 2.42 in Section 2.2.4 and Refs. [16] and [17]) considered fixed angular separations \( \varphi_{12} \) of \( \tilde{s}_1 \) and \( \tilde{s}_2 \), which samples all \( \omega \) since \( P_I(s_1, s_2, \varphi_{12}) \) is integrated over the nucleon (beam-direction) \( z \)-coordinates.

The angular correlation function \( \Gamma^I_{jj}(\omega) \) has little practical use in the calculation of cross sections as it couples the coordinates \( \hat{r}_1 \) and \( \hat{r}_2 \), preventing efficient factorization of integrals over these coordinates. However, all the dependence on \( I \) is encapsulated in \( \Gamma^I_{jj}(\omega) \), which depends only on the angular momenta of the two nucleons, their total angular momentum, and the angular separation of \( \hat{r}_1 \) and \( \hat{r}_2 \). It is therefore instructive to consider how \( \Gamma^I_{jj}(\omega) \) changes as \( I \) is varied.
4.1. ANGULAR CORRELATION FUNCTION IN JJ-COUPLING

Figure 4.1: Angular correlation function for the first direct term $\Gamma_{jj}^I(\omega)$ (Eq. 4.9) for a $[0d_{5/2}]^2$ configuration coupled to $I = 0$ (solid), 2 (dashed) and 4 (dotted). Since it is expected that larger $I$ lead to broader momentum distributions, we might conclude that larger angular separations lead to broader momentum distributions. If we compare to previous calculations for $P(s_1,s_2,\phi_{12})$ (Fig. 2 of Ref. [60]), the similarity in terms of how $\omega$ (or $\phi_{12}$) is probed is striking.

As an example we take pure $[0d_{5/2}]^2$ removal (i.e. $\beta_1 = \beta_2 = \beta'_1 = \beta'_2 = 0d_{5/2}$) for couplings $I = 0, 2$ and 4, relevant to sd-shell examples. The angular correlation function is shown in Fig. 4.1, encapsulating all effects of the total angular momentum $I$, and clearly depends strongly on $I$. We expect larger $I$ to lead to broader heavy residue momentum distributions and observe that since larger $I$ weight larger angular separations more strongly, larger $\omega$ should give the broader components of the residue momentum distributions.

Whilst we expect a strong dependence of the residue momentum distribution on the total angular momentum coupling, it is unclear whether, or how, the distribution depends on the orbital angular momentum of the nucleons removed. In Fig. 4.2 we consider the angular correlation function $\Gamma_{jj}^I(\omega)$ for $I = 0$ for the configurations $[\ell_{\ell+1/2}]^2$ with $\ell = 0 - 5$. For each configuration there is a peak at small angular separations, with $\Gamma_{jj}^I(\omega)$ being weakly dependent on $\omega$ for $\omega > 40^\circ$, and indeed being almost independent of the orbital angular momentum $\ell$, particularly so if one neglects $\ell = 0$. A similar example is considered in Fig. 4.3 for $I = 4$. The functional form is relatively independent of $\ell$ for $\omega < 50^\circ$ and certainly different to the $I = 0$ examples shown in Fig. 4.2. The different nucleon configurations will naturally have different radial wave functions $u_{\ell j}(r)$,
4.1. ANGULAR CORRELATION FUNCTION IN JJ-COUPLED

Figure 4.2: Angular correlation function $\Gamma_{jj}(\omega)$ for different $[\ell_{\ell+1/2}]^2$ configurations coupled to $I = 0$, showing $\ell = 0$ (solid), 1 (open circles), 2 (dotted), 3 (solid squares), 4 (dashed), and 5 (open triangles). $\Gamma_{jj}(\omega)$ is weakly dependent on both $\omega$ and $\ell$ for $\omega > 40^\circ$. The similarity for different values of $\ell$ suggests that the residue momentum distribution may be only weakly dependent on the two-nucleon configuration when compared to the sensitivity to $I$.

which to some extent determine how the angular separation is probed in the reaction, but the angular correlation functions themselves appear only weakly dependent on the nucleon configuration. This indicates the residue momentum distribution will be less sensitive to the nucleon pair configuration than the total angular momentum $I$ and, from a spectroscopic point of view, while they will give a strong signal of the spin of the final state, finer details of the shell model wave function will be more difficult to probe.

Note that the probing of $\omega$ by the direct and exchange terms does not necessarily follow the same form. For the case where $\alpha = \alpha'$ and $\beta_1 = \beta_2$, where $\beta_i$ denotes a particular orbital, we would expect the direct and exchange angular correlation functions to have the same functional form, though possibly a different phase. The examples considered here are all of this type. However, consider the case where $\alpha = \alpha'$, but $\ell_1$ is even and $\ell_2$ is odd. The parity Clebsch Gordan $(\ell_10\ell_1'0|k0)$ will ensure that for the direct terms the sum over $k$ will take only even values of $k$, but for the exchange terms only odd values of $k$, so the functional form of $\Gamma_{jj}(\omega)$ will be certainly be different for the direct and exchange terms.

The angular correlation function is similar in form to the two nucleon position prob-
4.2 Angular correlations in LS coupling

Here we express the angular correlation function Eq. 4.9 in LS coupling. The derivation is similar to and somewhat simpler than that for jj-coupling. We will see that it is the total orbital angular momentum $L$ that determines the probing of the two-nucleon angular separation and consequently the shape of the residue momentum distribution, and that the total angular momentum $I$ is of secondary importance. This will prove a useful tool for investigating the effect of the pair combination $\alpha$ on the residue momentum width, with examples discussed in Section 6.6.

Our starting point is the expression of the cross section in LS coupling (Eq. 2.105 of Section 2.3.5). Again considering the first direct term only we extract from this expression abilities $P_f(s_1, s_2, \varphi_{12})$ (Eq. 2.42) shown in Refs. [17] and [60] as a function of $\varphi_{12}$, the angle between $\hat{s}_1$ and $\hat{s}_2$. The present result $\Gamma_{jj}^I(\omega)$ arises from simple angular momentum algebra, whereas $P_f$ requires numerical integration (over the $z_i$ co-ordinates) of the two-nucleon overlap. In Ref. [60] the two nucleon position probability was displayed for the $S = 0$ and $S = 1$ components for various configurations and it is clear that the $S = 0$ and $S = 1$ components of $P_f(s_1, s_2, \varphi_{12})$ are rather different as a function of (in that case) $\varphi_{12}$, the angle between $\hat{s}_1$ and $\hat{s}_2$. We now express the angular correlation function $\Gamma_{jj}^I(\omega)$ in LS coupling.

Figure 4.3: Angular correlation function $\Gamma_{jj}^I(\omega)$ for different $[\ell_{\ell+1/2}]^2$ configurations coupled to $I = 4$, showing $\ell = 2$ (dotted), 3 (solid squares), 4 (dashed), and 5 (open triangles). The functional form of $\Gamma_{jj}^I(\omega)$ is weakly dependent on $\ell$. 
all quantities dependent on the projections of angular momenta, namely the four spherical harmonics from the single-nucleon wave functions Eq. 2.98, and two Clebsch-Gordan coefficients from the LS-coupled two-nucleon wave function, Eq. 2.97. We then evaluate the sums over \( \lambda \),

\[
\Gamma^L(\hat{r}_1, \hat{r}_2) = \sum_{\lambda_1 \lambda_2 \lambda_1' \lambda_2'} (\ell_1 \lambda_1 \ell_2 \lambda_2 | L \Lambda) (\ell_1' \lambda_1' \ell_2' \lambda_2' | L \Lambda) \nabla_{\ell_1 \lambda_1}(\hat{r}_1) \nabla_{\ell_1' \lambda_1'}(\hat{r}_1)^* \nabla_{\ell_2 \lambda_2}(\hat{r}_2) \nabla_{\ell_2' \lambda_2'}(\hat{r}_2)^*.
\]

(4.10)

By expanding the two Clebsch-Gordan coefficients using Eq. (3.11) of Ref. [46] we obtain

\[
\Gamma^L(\hat{r}_1, \hat{r}_2) = \sum_{\lambda_1 \lambda_2 \lambda_1' \lambda_2'} (-1)^{L+\ell_1-\ell_1'} \frac{L^2}{\ell_1'!} \sum_{k q} \hat{k} (\ell_1 \lambda_1 k q | \ell_1' \lambda_1') (\ell_2 \lambda_2 \ell_2' - \lambda_2' | k q) \nabla_{\ell_1 \lambda_1}(\hat{r}_1) \nabla_{\ell_1' \lambda_1'}(\hat{r}_1)^* \nabla_{\ell_2 \lambda_2}(\hat{r}_2) \nabla_{\ell_2' \lambda_2'}(\hat{r}_2)^* \mathbb{W}(\ell_1 \ell_2 \ell_1' \ell_2'; L k).
\]

(4.11)

We now sum over \( \lambda_i, \lambda_i' \) pairs and combine spherical harmonics of the same co-ordinate \( \hat{r}_i \), giving

\[
\Gamma^L(\hat{r}_1, \hat{r}_2) = \frac{\hat{L}^2 \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_1' \hat{\ell}_2'}{(4\pi)^2} (-1)^{L+\ell_1-\ell_1'} \sum_k \frac{1}{k^2} \mathbb{W}(\ell_1 \ell_2 \ell_1' \ell_2'; L k) (\ell_1 0 \ell_1' 0 | k 0) (\ell_2 0 \ell_2' 0 | k 0) \sum_q (-1)^q \mathbb{W}_{k q}(\hat{r}_1) \mathbb{W}_{k q}(\hat{r}_2),
\]

(4.12)

and define the *partial* angular correlation function \( \Gamma^L(\omega) \equiv \Gamma^L(\hat{r}_1, \hat{r}_2) \) as

\[
\Gamma^L(\omega) = (-1)^{L+\ell_1-\ell_1'} \frac{\hat{L}^2 \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_1' \hat{\ell}_2'}{(4\pi)^2} \sum_k \mathbb{W}(\ell_1 \ell_2 \ell_1' \ell_2'; L k) (\ell_1 0 \ell_1' 0 | k 0) (\ell_2 0 \ell_2' 0 | k 0) P_k(\cos \omega).
\]

(4.13)

The partial angular correlation function \( \Gamma^L(\omega) \) is then related to the full angular correla-
4.2. ANGULAR CORRELATIONS IN LS COUPLING

The partial angular correlation function by

\[
\Gamma_{LS}^I(\omega) = \sum_{SL} (-1)^{L+\ell_1-\ell'_1} \left\{ \begin{array}{ccc} \ell_1 & s & j_1 \\ \ell_2 & s & j_2 \\ L & S & I \end{array} \right\} \left\{ \begin{array}{ccc} \ell'_1 & s & j'_1 \\ \ell'_2 & s & j'_2 \\ L & S & I \end{array} \right\} \frac{\hat{\ell}_1 \hat{\ell}_2 \hat{\ell}'_1 \hat{\ell}'_2 \hat{j}_1 \hat{j}_2 \hat{j}'_1 \hat{j}'_2 \hat{L}^2 \hat{S}^2}{(4\pi)^2} \sum_k W(\ell_1 \ell_2 \ell'_1 \ell'_2; Lk) (\ell_1 0 \ell'_1 0 | k 0) (\ell_2 0 \ell'_2 0 | k 0) P_k(\cos \omega)
\]

\[
= \sum_{SL} \left\{ \begin{array}{ccc} \ell_1 & s & j_1 \\ \ell_2 & s & j_2 \\ L & S & I \end{array} \right\} \left\{ \begin{array}{ccc} \ell'_1 & s & j'_1 \\ \ell'_2 & s & j'_2 \\ L & S & I \end{array} \right\} \hat{S}^2 \hat{j}_1 \hat{j}_2 \hat{j}'_1 \hat{j}'_2 \Gamma^L(\omega),
\]

which is equivalent to the previous version in \(jj\) coupling of Eq. 4.9.

It is now apparent that the partial angular correlation \(\Gamma^L(\omega)\) is independent of \(I\) and that it is the total orbital angular momenta \(L\) that is crucial in determining how the angular separation is probed. In single-nucleon knockout it is the orbital angular momentum \(\ell\) that determines the residue momentum width, with larger \(\ell\) leading to wider distributions. The assumption here is that, in parallel with one-nucleon knockout, larger \(L\) leads to wider heavy residue momentum distributions. This is consistent with the result that large \(I\) lead to broader residue momentum distributions, since large \(I\) must necessarily probe larger \(L\). For each \(I\) several values of \(L\) may contribute, their relative strengths being determined by the 9\(j\)-coefficients and statistical factors of Eq. 4.14 and, structurally, where more than a single configuration \(\alpha\) contributions, by the two-nucleon amplitudes for a given transition. The relative weights of the \(L\) values will depend not only on the total angular momentum coupling \(I\), but also on the particular nucleon pair \(j_1\) and \(j_2\) and we may find cases of different nucleon pairs coupled to the same \(I\) that weight \(L\) very differently, thus giving different residue momentum distributions.

We show the partial angular correlation function in Fig. 4.4 for \(L = 0 \rightarrow 4\), for the case of \([d]^2\) removal. There is a strong sensitivity to the total orbital angular momentum \(L\). The functional form of \(\Gamma^L(\omega)\) is similar to the calculations of \(P_f(s_1, s_2, \varphi_{12})\) of Ref. [17] for \(S = 0\).

In the case where \(\ell_1 = \ell_2 = \ell'_1 = \ell'_2 = 0\) the only allowed value for \(k\) in the summation of Eq. 4.13 is \(k = 0\). This must also be true for the equivalent exchange term expression. Since \(P_0(\cos \omega)\) is independent of \(\omega\) the angular correlation function is then a constant and the corresponding cross section will be equivalent to the uncorrelated cross section, i.e. the fully correlated \([s_{1/2}]^2\) case is identical in shape to the uncorrelated case. The uncorrelated case will be discussed further in Section 6.1.

We will consider in Section 6.6 some examples of the angular correlation function, in
4.2. ANGULAR CORRELATIONS IN LS COUPLING

Figure 4.4: Partial angular correlation function $\Gamma^L(\omega)$ (Eq. 4.13) for $[d]^2$, coupled to $L = 0$ (solid), 1 (dashed), 2 (dotted), 3 (open circles) and 4 (open squares). We see a strong dependence of $\Gamma^L(\omega)$ on $L$, but unlike the case of $\Gamma^I_{jj}(\omega)$, one wouldn’t immediately come to the conclusion that larger angular separations lead to broader residue momentum distributions since $L = 3$ probes angles $\omega > 50^\circ$ more strongly that $L = 4$. Again we consider the calculations for $P_I(s_1, s_2, \varphi_{12})$ of Fig. 3 in Ref. [60]. The $S = 0$ contributions are explicitly displayed and for such cases $I = L$, and match the functional form of $\Gamma^L(\omega)$ once it is considered that a particular value of $\varphi_{12}$ will probe a range of $\omega$. 
4.3 Approximate momentum distributions with angular separation

We have seen that the angular correlation of the two-nucleons depends strongly on their coupling and that this is expected to determine the different heavy residue momentum distributions as $I$ is varied. The introduction of the angular correlation function $\Gamma_{jj}(\omega)$ prohibits the separation of integrals over $\vec{r}_1$ and $\vec{r}_2$ and so is not useful for practical calculations, but in principle $d\sigma/d\kappa_c$ should be sensitive to the values of the angular separation $\omega$ probed in the reaction. It is therefore of interest to explore this in some approximate fashion.

We calculate the residue momentum distribution $d\sigma/d\kappa_c$ for particular fixed values of $\omega$, by choosing very particular co-ordinates for nucleons 1 and 2. We consider the two nucleons at fixed and identical impact parameters $\vec{b}_1 = \vec{b}_2 = \vec{b}_v$ where $\vec{b}_v||\vec{b}_c$, such that $s_1 = s_2 = s$ as illustrated in Fig. 4.5. For such an arrangement, the co-ordinate $z_2$ can be written in terms of $z_1$ and the angular separation of the two nucleons $\omega$ by,

$$z_2 = s \tan(\tan^{-1}(z_1/s) - \omega). \quad (4.15)$$

With the co-ordinates constrained in this way we may (numerically) integrate over the $z_1$-co-ordinate, with the parameter $\omega$ determining the corresponding value of $z_2$, and so obtain the (approximate) residue momentum distribution as a function of the two-nucleon angular separation. As an example we consider the case where $\beta_1 = \beta_2$ and $\alpha = \alpha'$ so that the direct and exchange terms will have identical angular correlations and residue momentum distributions. The residue momentum distribution compromises the square modulus of an integral over the nucleon $z_1$-co-ordinate, with the second nucleon $z_2$-co-ordinate given by the angular separation of interest,

$$\frac{d\sigma}{d\kappa_c d\omega} \sim \int d\kappa_1 \int d\kappa_2 \delta(\kappa_c + \kappa_1 + \kappa_2) \Gamma_{jj}^I(\omega) \left| \int dz_1 u_{j\ell}(r_1) u_{j\ell}(r_2) \exp(i\kappa_1 z_1) \exp(i\kappa_2 z_2(z_1, \omega)) \right|^2. \quad (4.16)$$
In the following we have neglected the angular correlation function itself; for fixed angular separation it acts only as an absolute scaling factor. The objective here is to demonstrate that different angular separations give significantly different residue momentum distributions. The radial wave functions used are appropriate for $\pi[0d_5/2]^2$ removal from $^{28}$Mg and were calculated in a Woods-Saxon plus coulomb potential of standard geometry ($r_0 = 1.25$ fm, $a = 0.7$ fm, $V_{so} = 6$ MeV). Evaluating Eq. 4.16 numerically gives a residue momentum distribution dependent on the angular separation $\omega$ as shown in Fig. 4.6.

A simplistic calculation such as this cannot give a complete picture. The residue momentum distribution is expected to vary if $s_1 \neq s_2$. Since the target essentially samples a cylindrical volume near the surface of the projectile, the full reaction geometry and radial wave functions will constrain and sample $\omega$ in a rather complicated manner. The analysis presented here does however demonstrate the basic expectation that the width of the residue momentum distribution increases as the angular separation $\omega$ increases. However, the width seems to saturate at about $30^\circ$, after which it is approximately constant. This could be a feature of the approximate nature of the calculation; we have picked a very particular set of co-ordinates. What it would suggest is that, not so much that the broad components of the residue momentum distribution arise from large angular separations, but that the narrowest components arise from small angular separations, corresponding to some degree of spatial localisation and nucleon-pair spatial correlations.
4.3. MOMENTUM DISTRIBUTIONS WITH ANGULAR SEPARATION

Figure 4.6: Approximate heavy residue momentum distributions as a function of the angular separation of the two nucleons, given by Eq. 4.16. The lines show angular separations of $\omega = 0^\circ$ (solid), $5^\circ$ (open circles), $15^\circ$ (dotted), $30^\circ$ (open squares), $45^\circ$ (dashed) and $60^\circ$ (open triangles). The curves are normalised to the same peak value. The width increases very rapidly for small $\omega$ ($0^\circ - 30^\circ$), but then becomes narrower ($30^\circ - 60^\circ$). At the largest $\kappa_c$ the strongest contributions come from the largest $\omega$. 
Chapter 5

Approximations for $R_{\ell\lambda}^j$

The formalism of Chapter 2 has shown that the residue momentum distributions are built upon one-dimensional Fourier transforms (in the beam direction $z$) of the single nucleon wave function $\phi_{\ell j}(\vec{r})$, defined as $R_{\ell\lambda}^j(s,\kappa)$ (see Eqs. 2.86 and 2.93). The fully-correlated residue momentum distribution is then a complicated summation over the contributing $\lambda$ values, so some insight into the sensitivities of $R_{\ell\lambda}^j$ is useful. Here we take approximate forms for the radial part of the single nucleon wave function $u_{\ell j}(r)$ in order to evaluate the integral $R_{\ell\lambda}^j$ analytically, giving some insight into the factors controlling the single nucleon momentum distribution and thus the full heavy residue momentum distributions.

Ideally, we retain an explicit analytical dependence on $\lambda$ when evaluating $R_{\ell\lambda}^j$, since we have still to sum over $\lambda$ in order to obtain the full residue distribution. Classically, increasing alignment of the projection $\lambda$ of the orbital angular momentum $\ell$ corresponds to motion in the directions perpendicular to the beam direction. The nucleon therefore presents a larger cross sectional to the target and has relatively small momentum in the beam direction, corresponding to a larger cross section for large $\lambda$ states and particularly narrow longitudinal momentum distributions. This classical picture is discussed for the $\ell = 1$ case in Ref. [61].

5.1 Properties of $R_{\ell\lambda}^j$

Initially we consider the general properties of $R_{\ell\lambda}^j$, independent of any assumption about the radial wave function $u_{\ell j}(r)$. Firstly, it is essential that $R_{\ell\lambda}^j$ depends on $\lambda$, which enters solely through the associated Legendre polynomial. If $R_{\ell\lambda}^j$ is independent of $\lambda$ all terms in the summation of Eq. 2.93 have the same shape and the full residue momentum distribution will be independent of $I$ and thus independent of the final state populated. We also observe that the nucleon total angular momentum $j$ enters only through the nucleon radial wave function $u_{\ell j}(r)$, so that we might expect $R_{\ell\lambda}^j$ to be relatively insensitive to $j$
5.2. LARGE $\ell$ APPROXIMATIONS FOR $Y_{\ell\lambda}(\theta, \varphi)$

for a particular value of $\ell$ and $\lambda$. The value of $j$ is, however, important in determining which values of $\lambda$ contribute most strongly, through the angular momentum coupling factor $G_{\lambda_1 \lambda_2 \lambda_1' \lambda_2'}$.

We can rewrite the exponential in Eq. 2.86 to explicitly show the $z$–odd and even parts, i.e.

$$R_{\ell\lambda}^j(s, \kappa) = \frac{c_{\ell\lambda}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \ u_{j\ell}(r) P_{\ell}^{|\lambda|}(\cos \theta) \exp(\kappa z)$$

The terms $\cos(\kappa z)$ and $\sin(\kappa z)$ are clearly even and odd about the midpoint of the integral, $z = 0$. This point corresponds to an angle $\theta = \pi/2$ about which $P_{|\lambda|\ell}(\cos \theta)$ is even if $\ell + \lambda$ is even, and odd of $\ell + \lambda$ is odd. Since the integrand must clearly be even for the integral to be non-zero, we see immediately that if $\ell + \lambda$ is even $R_{\ell\lambda}^j$ will be purely real, and if $\ell + \lambda$ is odd $R_{\ell\lambda}^j$ is purely imaginary. We also note that the former, involving only $\cos(\kappa z)$, will be symmetric about $\kappa = 0$, and the latter involving only $\sin(\kappa z)$ will be asymmetric about $\kappa = 0$. We note that since $R_{\ell\lambda}^j$ is defined containing $P_{\ell}^{|\lambda|}(\cos \theta_i)$, the only difference between $R_{\ell\lambda}^j$ and $R_{\ell-\lambda}^j$ arises in the constant $c_{\ell\lambda}$.

We first extend these general considerations by using a large $\ell$ approximation for $Y_{\ell\lambda}(\theta, \varphi)$, applicable cases when $\lambda \ll \ell$. We then consider two approximations for $u_{\ell j}(r)$ in order to solve the integral over $z$ within $R_{\ell\lambda}^j(s, \kappa)$ analytically. We first approximate $u_{\ell j}(r)$ by a Gaussian type wave function, expected to reasonably describe the exact wave function near the nuclear surface. We then use a Hankel function in place of $u_{\ell j}(r)$, which has the correct asymptotic properties.
is

\[ Y_{\ell\lambda}(\theta, \varphi) \approx \frac{\exp(i\lambda \varphi)}{\sqrt{\sin \theta}} \sin \left[ (\ell + \frac{1}{2})\theta + (2\lambda + 1)\frac{\pi}{4} \right] \]

\[ = \frac{\exp(i\lambda \varphi)}{\sqrt{\sin \theta}} \left\{ \sin \left[(\ell + \frac{1}{2})\theta\right] \cos \left[(2\lambda + 1)\frac{\pi}{4}\right] + \cos \left[(\ell + \frac{1}{2})\theta\right] \sin \left[(2\lambda + 1)\frac{\pi}{4}\right] \right\}. \tag{5.2} \]

The trigonometric terms involving \((2\lambda + 1)\frac{\pi}{4}\) involve an odd number times \(\pi/4\), have an absolute value that is independent of \(\lambda\), and introduce only a phase,

\[ \cos \left[(2\lambda + 1)\frac{\pi}{4}\right] = (-1)^\alpha \sqrt{2}, \]

\[ \sin \left[(2\lambda + 1)\frac{\pi}{4}\right] = (-1)^\beta \sqrt{2}, \tag{5.3} \]

where \((-1)^\alpha = (-1)^\beta\) if \(\lambda\) is even and \((-1)^\alpha = -(-1)^\beta\) if \(\lambda\) is odd. Since the only term within \(R_{j\ell\lambda}\) dependent on \(\lambda\) is the associated Legendre polynomial, the only effect of \(\lambda\) is to change the relative phase of the terms \(\cos \left[(\ell + \frac{1}{2})\theta\right]\) and \(\sin \left[(\ell + \frac{1}{2})\theta\right]\). Since we can only either add or subtract these two terms there are only two shapes for \(R_{j\ell\lambda}\) from which the momentum distribution can be formed; those where \(\lambda\) is even and those where \(\lambda\) is odd.

We consider now the symmetry of the integrand of \(R_{j\ell\lambda}\) with respect to \(z = 0\). The terms \(\cos(\kappa z)\) and \(\sin(\kappa z)\) are even and odd about \(z = 0\) respectively, which corresponds to an angle \(\theta = \pi/2\). The radial wave function, depending on \(r\) only, is symmetric about \(z = 0\), so we consider

\[ \exp(i\kappa z)Y_{\ell\lambda}(\theta, \varphi) \approx \frac{\exp(i\lambda \varphi)}{\sqrt{2 \sin \theta}} [\cos(\kappa z) + i \sin(\kappa z)] \]

\[ \left\{ (-1)^\alpha \sin \left[(\ell + \frac{1}{2})\theta\right] + (-1)^\beta \cos \left[(\ell + \frac{1}{2})\theta\right] \right\}. \tag{5.4} \]

Whilst the large \(\ell\) approximation is not universally valid since the maximally aligned case \(\lambda = \ell\) will often be important, where it is valid we should see that \(R_{j\ell\lambda} \approx -R_{j\ell\lambda+2}\).

To illustrate the large \(\ell\) approximation we consider the example of \(\ell = 5\), \(\nu[0h_{11/2}]^2\) removal from \(^{136}\text{Xe}\), where the neutron separation is \(S_N = 7.4\) MeV. (Note that for this example, in an experiment the final residue states would also be populated indirectly, since \(S_p(\^{136}\text{Xe}) = 9.2\) MeV, but it will suffice for the present discussion). The required radial wave function was calculated in a Woods-Saxon well of standard geometry \(r_0 = 1.25\) fm and \(a = 0.7\) fm, with the depth adjusted to reproduce the experimental separation energy. We show the calculations for the exact and large-\(\ell\) spherical harmonics in Figs. 5.1 and
5.3 Gaussian wave function

We expect most two-nucleon removal events to occur when the nucleon is located near the projectile surface (see Fig. 2.4, \(^{28}\text{Mg}(-2p)\)). Provided the gross features of the wave function are reasonably described, the incorrect asymptotics of a Gaussian type wave function may be unimportant. In many cases we are interested in well-bound nucleon removal, where the contributions from the tail are in any case small. We take our nucleon radial wave function to be of the following Gaussian form,

\[ u_\ell(r) = N(\ell) \exp(-\frac{1}{2}\alpha^2 r^2)r^\ell, \]  

(5.5)

allowing us to calculate the integrals \( R_{\ell \lambda}(s, \kappa) \) for particular values of \( \ell \) and \( \lambda \). This simple radial wave function is independent of \( j \), so we neglect this label, writing instead \( R_{\ell \lambda} \). \( N(\ell) \) is the \( \ell \)-dependent normalisation constant of the radial wave function, which

Figure 5.1: Exact calculations of \( R_{\ell \lambda}(s = 8, \kappa) \) for \( \ell = 5 \) and \( \lambda \) values 0 (solid), 2 (dotted) and 4 (dashed), for two neutron removal from \(^{136}\text{Xe}\). The open circles show results using the large \( \ell \) approximation to \( Y_{\ell \lambda}(\theta, \varphi) \).

5.2 for even \( \lambda \) and odd \( \lambda \) respectively. As expected, the large \( \ell \) approximation reproduces the small \( \lambda \) distributions best and almost exactly in the \( \lambda = 0 \) and \( \lambda = 1 \) cases.
for $\ell = 2$ is

$$N(2) = \left[ \frac{16\alpha^7}{15\pi} \right]^{1/2}.$$  \hspace{1cm} (5.6)

$\alpha$ is an overall size parameter, scaling with the mass and size of the projectile. The size parameter is taken from Ref. [62], where

$$\hbar \omega = 45A^{-1/3} - 25A^{-2/3} \text{ MeV},$$ \hspace{1cm} (5.7)

$$\alpha = \left( \frac{m\hbar \omega}{\hbar^2} \right)^{1/2}.$$ \hspace{1cm} (5.8)

$\alpha$ decreases as the projectile mass increases ranging from 0.6 for $^{12}\text{C}$ to 0.40 for $^{208}\text{Pb}$. With this approximation for $u_\ell(r)$, the integral to solve is then

$$\mathcal{R}_{\ell\lambda} = \frac{C_{\ell\lambda}}{\sqrt{2\pi}} N(\ell) \exp\left( -\frac{1}{2} \alpha^2 s^2 \right) \int_{-\infty}^{+\infty} dz \exp(i\kappa z) \exp\left( -\frac{1}{2} \alpha^2 z^2 \right) r^\ell P_\lambda^\ell(\cos \theta).$$ \hspace{1cm} (5.9)

It is not clear how such an integral may be solved in general, but it is solvable analytically for particular values of $\ell$ and $\lambda$, and here we explicitly consider the $\ell = 2$ case. The simple analytic form for $P_\ell^\ell(\cos \theta)$ allows the general maximally aligned $\lambda = \ell$ case for to be found.
5.3.1 Explicit case of $\ell = 2$

The $\ell = 2$ case is particularly important for many $sd$-shell examples of two-nucleon knockout, which are often dominated by the $d_{5/2}$ and $d_{3/2}$ shells e.g. $^{22}\text{Mg}(-2n)$, $^{38}\text{Si}(-2p)$ and $^{28}\text{Mg}(-2p)$ discussed in Section 7.1. Here we use integrals from Gradshteyn and Ryzhik [63] to obtain

$$
\mathcal{R}_{20}(s, \kappa) = X_0(s) \exp(-\kappa^2/2\alpha^2) \frac{(\alpha^2 - \alpha^4 s^2/2 - \kappa^2)\sqrt{2\pi}}{\alpha^5},
$$

$$
\mathcal{R}_{21}(s, \kappa) = X_1(s) \exp(-\kappa^2/2\alpha^2) \frac{6i\sqrt{\pi} s \kappa}{\alpha^3 \sqrt{2}},
$$

$$
\mathcal{R}_{22}(s, \kappa) = X_2(s) \exp(-\kappa^2/2\alpha^2) \frac{3\sqrt{2\pi} s^2}{\alpha},
$$

(5.10)

where the $s$ dependent coefficient $X_\lambda(s)$ is common and is

$$
X_\lambda(s) = \frac{C_{2\lambda}}{\sqrt{2\pi}} \left[ \frac{16\alpha^7}{15\pi} \right]^{1/2} \exp(-\frac{1}{2}\alpha^2 s^2).
$$

(5.11)

The shapes of the momentum distributions are essentially Gaussian, $\exp(-\kappa^2/2\alpha^2)$, with width determined by $\alpha$. Larger mass projectiles lead to smaller $\alpha$ i.e. a more diffuse nucleon wave function and narrower intrinsic momentum distribution. This reflects the uncertainty principle as the nucleon wave function becomes more extended. Classically, alignment of the angular momentum with respect to the beam direction $z$ means the nucleon orbits in a plane perpendicular to the beam direction and accordingly the additional factors $\kappa^2$ and $\kappa$ in the $\lambda = 0$ and 1 cases increase the width of the distribution as this plane is inclined towards the beam direction. Note that the $\lambda = 0$ and 2 cases are symmetric about $\kappa = 0$, whereas the $\lambda = 1$ case is asymmetric, as expected by simple symmetry arguments.

As $s$, the distance from the core in the impact parameter place at which the nucleon is probed, increases, the Gaussian radial wave function suggests the width of $\mathcal{R}_{2\lambda}$ to be constant for $\lambda = 1$ and 2. The shape of the $\lambda = 0$ case is determined by the relative strengths of the terms in $(\alpha^2 - \alpha^4 s^2/2 - \kappa^2)$. If the first two terms are larger in magnitude than the last term, $\mathcal{R}_{20}$ will have similar shape to $\mathcal{R}_{22}$. For most cases $\alpha^2 \approx 0.25$ and $\kappa^2 \approx 1$. For small $s$, the $\kappa^2$ term dominates, giving a relatively broad distribution. For $s > 6$ the $\alpha^4 s^2/2$ term will dominate and we would expect the $\mathcal{R}_{20}$ distribution to approach that for $\mathcal{R}_{22}$ with the two being asymptotically identical (in as far as we can draw conclusions from the asymptotically incorrect Gaussian wave function).

Whilst we have derived simple analytic forms for $\mathcal{R}_{\ell\lambda}$ explicitly for $\ell = 2$, the functional forms for each value of $\lambda$ are different and it is difficult to see how one might produce
5.3. GAUSSIAN WAVE FUNCTION

a full heavy-residue momentum distribution; as this involves a sum over four \( \lambda \) values weighted by the appropriate Clebsch-Gordon coefficients.

5.3.2 General case for \( \lambda = \ell \)

Here we consider the case where \( \lambda = \ell \), the maximally aligned case. Classically, we might expect the cross section for the maximally aligned case to be the largest, since the nucleon orbits perpendicular to the beam direction and presents the largest cross sectional area to the target (see e.g. Ref. [61]). The associated Legendre polynomial has the rather simple form. Taken from Abramowitz and Stegun 8.6.16 [57], the maximum projection associated Legendre polynomial is

\[
P_{\ell\ell}^{(\cos \theta)} = (-1)^{\ell}(\sin \theta)^{\ell}(2\ell - 1)!!.
\]

The integral \( \mathcal{R}_{\ell\ell} \) for a Gaussian radial wave function then becomes

\[
\mathcal{R}_{\ell\ell}(s, \kappa) = \frac{C_{\ell\ell}}{\sqrt{2\pi}} N(\ell) \exp\left(-\frac{1}{2} s^2\alpha^2\right) \int_{-\infty}^{+\infty} dz \exp(i\kappa z) \exp\left(-\frac{1}{2} \alpha^2 z^2\right) r^{\ell} (-1)^{\ell}(\sin \theta)^{\ell}(2\ell - 1)!!.
\]

By virtue of the fact that \( (\sin \theta)^{\ell} = s^{\ell}/r^{\ell} \) the integral is much simplified, and by again using the integrals from Ref. [63] we obtain the result for the maximally aligned case,

\[
\mathcal{R}_{\ell\ell}(s, \kappa) = \frac{C_{\ell\ell}}{\alpha} N(\ell) \exp\left(-\frac{1}{2} \alpha^2 s^2\right)(-1)^{\ell}(2\ell - 1)!! \ s^{\ell} \exp\left(-\frac{\kappa^2}{2\alpha^2}\right).
\]

The width of the fully aligned \( \lambda = \ell \) component is independent of \( \ell \) and depends only on the projectile range parameter \( \alpha \). As expected from the symmetry properties of \( P_{\ell}^{(\cos \theta)} \), the fully aligned distribution is symmetric about \( \kappa = 0 \).

5.3.3 Full residue distribution for the maximally aligned case

It is difficult to take the Gaussian approximation further to obtain a full residue momentum distribution, since we must still sum over all \( \lambda \) weighted by the appropriate Clebsch-Gordon coefficients. The one case where we can proceed is that of the maximally aligned case (for the maximum total angular momenta), i.e. \( \lambda_1 = \lambda_2 = \lambda'_1 = \lambda'_2 = 2 \). We then need only consider \( \mathcal{R}_{22} \), specifically, the product of four such functions. Neglecting
all but terms dependent on $\kappa_i$, the integral

$$
\frac{d\sigma_{\text{max}}}{d\kappa_c} \propto \int_{-\infty}^{+\infty} d\kappa_1 \int_{-\infty}^{+\infty} d\kappa_2 \delta(\kappa_c + \kappa_1 + \kappa_2) \left| \exp(-\kappa_1^2/2\alpha^2) \exp(-\kappa_2^2/2\alpha^2) \right|^2 
$$

$$
\propto \exp(-\kappa_1^2/\alpha^2) \int_{-\infty}^{+\infty} d\kappa_1 \exp(-(2\kappa_1^2 + \kappa_c\kappa_1)/\alpha^2) 
$$

$$
\propto \exp(-\kappa_c^2/2\alpha^2),
$$

(5.15)

should be characteristic of the maximally aligned $I = 4$, $\mu = 4$. The last line is obtained by using Eq. 3.323.2 of Ref. [63]. We compare this result to the full calculations in Section 6.2.

### 5.4 Hankel wave function

Here we consider an expression for the radial wave function $u_{\ell j}(r)$ that has the correct asymptotic form, namely a Hankel function, appropriate for neutron wave functions. Lo Monaco and Brink [64], and Johnson [65] derive the one-dimensional ($z$) Fourier transform of $\Phi_{\ell m}(\vec{r}) = h_1^{(1)}(i\gamma r)Y_{\ell m}(\theta, \phi)$. The Hankel function has the correct asymptotic form for the radial wave function, whereas the Gaussian wave function used previously does not. However, it describes inaccurately the interior of the wave function. The decay constant of the Hankel function $\gamma$ is related to the nucleon separation energy $S_N$ by

$$
\gamma = \sqrt{\frac{2\mu S_N}{\hbar^2}},
$$

(5.16)

where $\mu$ is the reduced mass. For neutrons removed in $^{22}\text{Mg}$, where the neutron separation is $S_n = 17$ MeV, $\gamma = 0.88$ fm$^{-1}$. LoMonaco and Johnson’s starting point uses the full spherical harmonic, whereas we have already extracted the $\phi$ dependent part $\exp(i\lambda \varphi)$ in our definition of $R_{\ell \lambda}$. We also include a factor $1/\sqrt{2\pi}$ in our definition of $R_{\ell \lambda}$. The result they obtain is

$$
\tilde{\Phi}_{\ell \lambda}(\vec{s}, \kappa) = \sqrt{\frac{2\pi}{\gamma^2}} \exp(i\lambda \varphi) R_{\ell \lambda}(s, \kappa) 
$$

$$
\tilde{\Phi}_{\ell \lambda}(\vec{s}, \kappa) = \int_{-\infty}^{+\infty} dz \exp(i\kappa z) \Phi_{\ell \lambda}(\vec{r}) 
$$

$$
= -\frac{2}{\gamma^2} Y_{\ell \lambda}(\beta, \varphi) K_{\lambda}(\eta s),
$$

(5.17)

where $\eta$ is defined as

$$
\eta = \sqrt{\gamma^2 + \kappa^2}.
$$

(5.18)
5.4. HANKEL WAVE FUNCTION

The minimum value of η comes when κ = 0 such that η = γ. The angle β of the spherical harmonic in Eq. 5.17 is complex and is defined such that its cosine and sine are given by

\[
\cos \beta = \frac{iκ}{\sqrt{η^2 - κ^2}} = \frac{iκ}{γ}, \quad (5.19)
\]

\[
\sin \beta = \frac{η}{\sqrt{η^2 - κ^2}} = \frac{η}{γ}. \quad (5.20)
\]

Note that the cosine of β is purely imaginary and the sine is purely real. This requires β to have a particular real part \(\Re(β) = \frac{(2n+1)\pi}{2}\), such that \(β = \frac{(2n+1)\pi}{2} + iβ\), where n is an arbitrary integer. Spherical harmonics with complex angles require careful treatment, in particular in complex conjugation. The definition of the spherical harmonic in terms of a polynomial expansion is useful for calculation when the angle is complex. When it is required it may be evaluated using a polynomial expansion of the spherical harmonic [66],

\[
Y_\ell\lambda(β, ϕ) = \exp(iλϕ) \sum_{pqmn} \frac{(-1)^p}{p!q!n!} 2^{p+q} \left[ \frac{η^{\ell-n}κ^n}{γ^\ell} \right], \quad (5.21)
\]

where the sums over p, q and n are constrained such that \(p + q + n = \ell\) and \(p - q = \lambda\).

The Hankel function approximation for \(u_\ell(r)\) is exact in the asymptotic region, so we should naturally consider Eq. 5.17 as s becomes large. The asymptotic form for the modified spherical Bessel function [57] is

\[
K_\lambda(ηs) \to \sqrt{\frac{π}{2ηs}} \exp(-ηs). \quad (5.22)
\]

which is independent of λ. The spherical harmonic is essentially a polynomial in κ and γ, and will increase rapidly for κ > 1. The values of κ are curtailed by the modified spherical Bessel function \(K_\lambda(ηs)\) which decreases exponentially as κ increases due to increased η. Larger s leads to a faster decaying exponential and a more severe attenuation towards large κ, leading to narrower distributions if the nucleon is removed in more peripheral regions.

The Hankel function explicitly contains the binding energy of the nucleon in γ. As the nucleon separation energy \(S_N\) increases, the parameter γ becomes larger. This increases the rate of decay of the exponential Hankel function, so we expect that, relative to a more weakly bound system, smaller values of s will be emphasised. From the spherical harmonic Eq. 5.21, the γ dependent part is \(η^{\ell-n}/γ^\ell = (κ^2 + γ^2)^{(\ell-n)/2}/γ^\ell\). If γ is larger, this factor will vary more slowly over a given range of κ, giving a broader single-nucleon momentum distribution.
5.5 Comparisons of approximations for $R_{2\lambda}^j$

Here we compare the approximations for $R_{2\lambda}$ with the exact calculations using a wave function in a Woods-Saxon potential with standard geometry parameters $r_0 = 1.25$ and $a = 0.7$. The relevant radial wave functions themselves are shown in Fig. 5.3, where the Hankel function was normalised to the Woods-Saxon wave function at 20.0 fm. The Hankel function matches the Woods-Saxon exactly for $r \geq 5$ and the Gaussian reasonably describes its gross features for $r \leq 6$. Note that the projection of $s$, being the projection of $\vec{r}$ on the $xy$ plane, represents the minimum value of $r$ probed for the approximation of $R_{\ell \lambda}(s, \kappa)$, so the accuracy of $u_{j\ell}(r)$ at $r = s$ is only an indicator of the accuracy of $R_{\ell \lambda}$.

The calculations are shown in Fig. 5.4 for $\lambda = 0$, 1 and 2, for $s = 2$, 4, 6 and 8 fm. As expected, the Gaussian is in better agreement at small $s$ and the Hankel in perfect agreement in the asymptotic region. Calculations using the shell-model wave function show that the knockout events arise, for the example of $^{28}\text{Mg}(−2p)$, from $s = 2 − 6$ fm (see Fig. 2.4) and we would expect a similar probing in the present case of $^{22}\text{Mg}(−2n)$. Despite the simplicity of the Gaussian wave function, it describes $R_{\ell \lambda}(s, \kappa)$ well over this region, whereas the Hankel approximation strongly overestimates the importance of small
5.5. COMPARISONS OF APPROXIMATIONS FOR $R_{2\lambda}$

Figure 5.4: Comparison of Hankel (dotted), Gaussian (dashed) and exact (solid) calculations of $R_{\ell\lambda}(s, \kappa)$ for $\ell = 2$ and projections $\lambda$ (rows) and values of $s$ (columns), appropriate for $^{22}\text{Mg}(−2n)$. Note the rapid drop in the vertical scale as $s$ increases (shown atop each column). The shape of the $\lambda = 1$ and 2 distributions become somewhat narrower as $s$ increases, and the $\lambda = 0$ shape becomes similar to that of $\lambda = 2$ for large $s$. For $s \geq 6$ the Hankel function and Woods-Saxon calculations are essentially identical.
Whilst it is clear that a fraction of the cross section arises from events interacting with the tails of the nucleon wave function, a significant proportion arise from less peripheral events and the use of a Hankel wave function would lead to a large overestimation at small $s$, most likely giving a broader residue distribution. The predictions by the Gaussian wave function on the dependence on $s$ are borne out by the full calculations. We see that the width of the distributions for $\lambda = 1$ and 2 are only weakly dependent on $s$ and that the $\lambda = 0$ case becomes similar to the $\lambda = 2$ case as $s$ increases.
Chapter 6

Momentum distribution sensitivities

Residue longitudinal momentum distributions following single nucleon removal have a width characteristic of the orbital angular momentum of the removed nucleon, a signature exploited in numerous experiments with light exotic nuclei (e.g. Refs.[1, 47, 67]). In addition to the angular momentum content, they show a degree of sensitivity to the nucleon binding and reaction peripherality [5, 6, 61, 68]. Here we use the complete formalism developed in Chapter 2 to, building on the approximations discussed in Chapters 4 and 5, consider such sensitivities for the more complex case of two-nucleon removal. Unlike eikonal models of single-nucleon removal, the cross section does not factorise into structural (spectroscopic factor) and reaction dynamics parts, so the dependence on the nucleon quantum numbers is less transparent. We will show that, consistent with earlier, simple estimates [16, 17], the total angular momentum of the two nucleons is the main factor determining the width of the distributions. The residue momentum distributions can thus be exploited for final-state spin assignments. For this reason the total angular momentum is of practical importance, but the discussion of angular correlations in Chapter 4 showed the essential parameter controlling the width of the residue distribution is the total orbital angular momentum $L$. This leads naturally into discussions of the sensitivity to different configurations, that is nucleon angular momenta $\ell$ and $j$. We also consider the sensitivity to the nucleon binding energy and reaction mechanism (stripping or diffractive-stripping), and final state alignment effects.

Each sensitivity is illustrated with an example, taken to be two-proton removal from $^{28}$Mg at 82 MeV/nucleon on a beryllium target, typically considering only the $[0d_{5/2}]^2$ removal configuration (see Section 7.1.3 for details of wave functions and $S$-matrices). The consideration of a single configuration greatly simplifies the calculation and related discussions. In physical examples there are often many two-nucleon configurations (denoted $\alpha$ in Eq. 2.22) that contribute, consisting of a pair of orbitals $\beta_1$ and $\beta_2$ weighted by a signed two-nucleon amplitude $C_{\alpha}^{I_f}r_{\beta_1\beta_2}^{I}$, and as such are usually more complex than
6.1 Nucleon pair total angular momentum

In single-nucleon knockout the orbital angular momentum is the primary factor in determining the width of the residue momentum distribution, with larger values leading to broader residue momentum distributions for a given separation energy. The analogue in two-nucleon knockout is the total angular momentum of the two nucleons $I$, which must also couple the initial and final states $J_i$ and $J_f$. Previous work [16, 17] has shown there is a strong sensitivity to $I$ (based on calculations for fixed projections of the core-valence co-ordinate on the impact parameter plane $s_1$ and $s_2$) and in Section 4.1 we saw that the total angular momentum strongly alters the angular correlation of the two nucleons.

To date, all cases studied experimentally have involved an even-even projectile, the ground state spin of which is necessarily zero, so that final state spin $J_f$ is equal to the total angular momentum of the orbital pair $I$. This necessarily means only a single $I$ can contribute, and a strong dependence of the heavy residue momentum distribution on nucleon coupling $I$ would allow us to identify the spin of the final states populated. This information is invaluable information for providing extensive structural tests in exotic nuclei. Cases where the projectile spin is non-zero will be more complicated. Several $I$ values can contribute to each final state, the strength of each being largely determined by the relevant two-nucleon amplitudes, so the width for each final state will be less distinct.

We illustrate the dependence on $I$ by considering pure $\pi[0d_{5/2}]^2$ removal from $^{28}$Mg, shown in Fig. 6.1. The width increases with the total angular momentum with each state having a distinguishably different width. The full-width-half-maxima of the the $I = 0$, 2 and 4 states are 200, 280 and 480 MeV/c respectively - a factor of 2.4 increase in width from $I = 0$ to $I = 4$. Intriguingly, the width of the $I = 2$ distribution is much closer to that of the $I = 0$ distribution than that of $I = 4$. We return to this point in Section 6.6.

The $I = 0$ distribution (FWHM≈200 MeV/c) is particularly narrow and even significantly narrower than the distribution resulting from single $\pi[0d_{5/2}]$ removal (280 MeV/c) from the same system. The $\pi[1s_{1/2}]$ single-nucleon removal distribution from $^{28}$Mg has FWHM≈120 MeV/c. The uncorrelated case, calculated using Eq. 2.112, is also shown in Fig. 6.1 and does not accurately match the fully-correlated calculations for any of the final states, indicating the importance of the angular correlations induced by proper angular momentum coupling.

The essential differences when comparing the uncorrelated limit to the fully correlated calculations is that the angular momentum coupling of the two nucleons is neglected. In addition, we neglect antisymmetrization and restrict the projection of the orbital an-
Figure 6.1: $^{26}\text{Ne}$ residue momentum distributions for different pair couplings for pure $\pi[0d_{5/2}]^2$ removal from $^{28}\text{Mg}$, showing $I = 0$ (solid), 2 (dotted) and 4 (dashed). Also shown is the $[0d_{5/2}]$ single-proton knockout distribution (open circles) and the uncorrelated two-nucleon removal distribution (open squares). All curves have been normalised to the same peak value. The fully correlated calculations show an increase in width as $I$ increases, none of which is well described by the uncorrelated calculations. The $I = 0$ case is significantly narrower than the single-nucleon case, which is of similar width to the $I = 2$ case.
6.2 Final state alignment

The alignment is a measure of the orientation of the final state angular momentum with respect to the z-axis (beam direction). For single-nucleon knockout it is expected that the final states populated should be somewhat aligned, that is, the reaction mechanism will preferentially populate final states with larger projections of the final state angular momentum in the beam direction (see e.g. Fig. 12 of Ref. [1]). Classically, the larger projections correspond to the nucleon orbiting in a plane perpendicular to the beam
direction, such that when faced with a probe moving in the beam direction they present a larger cross sectional area than nucleons orbiting in a plane that lies in the beam direction. This results in a larger cross section for aligned final states. Such orbits, with the nucleon motion being largely confined to a plane perpendicular to the beam direction, also result in narrow residue momentum distributions (see Fig. 11 of Ref. [61], Ref. [6]). A first experimental observation of this predicted effect was recently observed in the angular distribution of \(\gamma\)-rays following proton knockout from \(^{32}\)Al [69].

Alignment effects have previously been observed in peripheral fragmentation reactions. Ref. [70] observe significant alignment of the high spin \(19/2^-\) isomer in \(^{53}\)Sc populated via fragmentation of \(^{46}\)Ti. Such reactions, where two-neutrons and a proton are removed, are beyond the scope of current discussions, and could contain strong contributions from mechanisms other than a single-step direct reaction. Nevertheless, a similar effect is expected in two-nucleon knockout and was searched for by Ref. [71] in \(\gamma-\gamma\) coincidences following two-proton removal from \(^{32}\)Mg. In Fig. 6.3 we show the \(\mu\)-substate decomposition of a \(4^+\) state in \(^{26}\)Ne populated via pure \([0d_5/2]^2\) proton removal from \(^{28}\)Mg. The reaction has a tendency to produce aligned final states - the fractional population increases from 8.6\% for \(\mu = 0\) to 17.1\% for \(\mu = 4\). Increased alignment gives to narrower contributions to the residue momentum distribution, with full width half maxima (FWHM) of 600 MeV/c for the \(\mu = 0\) case and 260 MeV/c for the \(\mu = 4\) case. The FWHM of the total \(I = 4\) cross section is \(\approx 480\) MeV/c. It is clear that restricting the residue momentum to values near the central value will lead to a large degree of alignment of the residue. We discuss this alignment effect for high spin final states in the case of \(^{208}\)Pb\((-2p)\) in Section 7.2.

Regardless of \(I\), we expect the maximally aligned state to i) be preferentially populated, and ii) have the narrowest residue momentum distribution. In Fig. 6.4 we compare \(d\sigma/d\kappa_c\) distributions for the maximally aligned components of the \(I = 0, 2, 4\) states resulting following pure \(\pi[0d_{5/2}]^2\) removal from \(^{28}\)Mg, showing similar widths in all three cases. We also compare the Gaussian approximation for the maximally aligned case Eq. 5.15, discussed in Section 5.3, which reproduces the expected residue momentum distribution width rather well.

The primary result of such alignment effects will be non-isotropic emission of \(\gamma\)-rays from the reaction residues. Ref. [69] observed this effect for the case of single-proton knockout from \(^{22}\)Mg and the \(\gamma\)-ray the angular distribution was shown to be sensitive to cuts on the residue longitudinal momentum. In principle \(\gamma\)-ray angular distributions and \(\gamma-\gamma\) angular correlations can be used to deduce the multipolarity of the \(\gamma\)-transition. The projectile alignment will also be important in two-step fragmentation reactions for
Figure 6.3: Decomposition of the $^{26}\text{Ne}(J^p = 4^+)$ residue longitudinal momentum distribution resulting from pure $[0d_{5/2}]^2$ knockout from $^{28}\text{Mg}$ into contributions from the different magnetic substates. Shown are $\mu = 0$ (solid), $|\mu| = 1$ (dashed), 2 (dotted), 3 (dot-dashed) and 4 (dot-dot-dashed), and the total (open circles). The percentage (momentum-inclusive) populations are 8.6, 17.3, 18.1, 22.0 and 34.2 % for $|\mu|=0–4$ respectively. The substate momentum distributions narrow as the alignment increases. The full width half maxima (FWHM) vary greatly for the different final state projections, from $\sim$600 MeV/c for $\mu=0$ to $\sim$260 MeV/c for the maximally aligned $\mu=4$. The FWHM for the total distribution is $\sim$480 MeV/c.
Figure 6.4: $^{26}\text{Ne}(I\pi)$ residue momentum distributions for the maximally aligned case ($\mu = I$) for $I = 0$ (solid), 2 (dashed) and 4 (dotted) from $|0d_{5/2}|^2$ proton removal from $^{28}\text{Mg}$, normalised to the same peak value. We also show the Gaussian wave function approximation for the maximally aligned case Eq. 5.15 (open circles). The distribution width varies rather little for different $I$ and the Gaussian wave function is shown to give a reasonable approximation.
secondary beams where the ground state has \(J_i \neq 0\). Further, in even-even projectile cases, if the projectile possesses isomeric states, it is likely to be somewhat aligned (though perhaps more pressing would be the population of the isomeric state itself, most likely giving significantly different final-state branching ratios).

### 6.3 Nucleon binding energy

We consider the degree of sensitivity of the two-nucleon stripping residue momentum distributions to the two-nucleon binding energy. In one-nucleon knockout it is well-documented that stronger binding leads to more spatially confined wave functions and to broader momentum distributions, a reflection of the uncertainty principle (see e.g. Refs. [5] and [1]). Given the importance of pair correlations, as already discussed, the resulting \(\kappa_c\) distributions are less intuitive in two-nucleon removal. The separation energies of well-bound nucleons are typically of the order of 15-20 MeV in cases of recent interest (and in later examples) and were 15 MeV per proton in the \(^{28}\text{Mg}\) test case used above. Figure 6.5 shows results of fully-correlated calculations of the longitudinal momentum distributions for \(I^* = 0^+\), \(\pi [0d_{5/2}]^2\), two-proton stripping from \(^{28}\text{Mg}\) at 82 MeV/nucleon. The curves shown are for protons initially bound by 1, 2, 5, 10, 15, and 20 MeV. The momentum distributions are subtly changed with the width increasing with the binding energy, but are nevertheless rather insensitive to the proton separation energy, particularly considering the range of binding energies covered. Within a particular example, the nucleon binding energies will vary by at most a few MeV due to the different final residue state excitation energies.

### 6.4 Diffractive-stripping contributions

The diffractive-stripping cross section is complicated by the projection-off of bound states for the elastically scattered nucleon. The diffraction operator \(\mathcal{K}_{ds}\) is divided into two parts, \(\mathcal{K}_2\) which is similar in structure to the pure stripping terms, and \(\mathcal{K}_3\) which removes contributions elastically scattering between bound states. In Section 2.2.6 we rewrote the diffraction operator to emphasizes the expected surface localisation of the diffractive cross section, effectively replacing \(S_i \rightarrow 1 - S_i\) for the diffracted nucleon, giving \(\tilde{\mathcal{K}}_2\) and \(\tilde{\mathcal{K}}_3\). This replacement is useful for convergence of the integral over the center-of-mass impact parameter \(b\) and greatly reduces the significance of the projection-off terms, as shown in Table 6.1 for the example of pure \(\pi [0d_{5/2}]^2\) removal from \(^{28}\text{Mg}\). The correction falls from 99% to just 15%. Fig. 6.6 shows the cross section as a function of the center-of-mass
6.4. DIFRACTIVE-STRIPPING CONTRIBUTIONS

Figure 6.5: Sensitivity of the residue longitudinal momentum distribution following two-proton removal from $^{28}\text{Mg}$ to the separation energy of the removed nucleons, showing (single-nucleon) binding of 1 (solid), 2 (open circles), 5 (dotted), 10 (solid squares), 15 (dashed) and 20 MeV (open triangles), with all curves normalised to the same peak. Within a given physical example of an inclusive cross section the effective separation energy varies due to the different residue final states. The variation in the (single-nucleon) effective separation energy is typically 2 MeV. This considered, we would expect to see negligible variation due to changes in binding within a given example.
Table 6.1: Diffractive-stripping cross sections $\sigma_{ds}^{(2)}$ and $\sigma_{ds}^{(3)}$ arising from different parts of the total diffractive-stripping operator $K_{ds}$. The two arrangements of the operator are shown, $S_i$ and $(1 - S_i)$, corresponding to $K_i$ and $\bar{K}_i$ respectively. $F_{po}$ is the percentage of $\sigma_{ds}^{(2)}$ removed by $\sigma_{ds}^{(3)}$. For the original arrangement, the projection-off term removes $\approx 99\%$ of the first term, whereas with the rearrangement the projection-off is just $15\%$. For comparison, the single-proton $\pi[d_{5/2}]$ stripping cross section for the same system is $\sigma_{1p} = 11.66$ mb, consistent with the expectation that the cross section $\sigma_{ds}^{(2)}$ arising from $K_2$ should be of the order of twice the single-nucleon stripping cross section.

<table>
<thead>
<tr>
<th>State</th>
<th>Operator</th>
<th>$\sigma_{ds}^{(2)}$</th>
<th>$\sigma_{ds}^{(3)}$</th>
<th>$\sigma_{ds}$</th>
<th>$F_{po}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0$^+$</td>
<td>$S_i$</td>
<td>15.924</td>
<td>15.635</td>
<td>0.289</td>
<td>98</td>
</tr>
<tr>
<td></td>
<td>$(1 - S_i)$</td>
<td>0.342</td>
<td>0.053</td>
<td>0.289</td>
<td>15</td>
</tr>
<tr>
<td>4$^+$</td>
<td>$S_i$</td>
<td>16.238</td>
<td>16.096</td>
<td>0.142</td>
<td>99</td>
</tr>
<tr>
<td></td>
<td>$(1 - S_i)$</td>
<td>0.168</td>
<td>0.025</td>
<td>0.142</td>
<td>15</td>
</tr>
</tbody>
</table>

impact parameter for different diffraction operators $K_2$ and $\bar{K}_2$. We see that $\bar{K}_2$ probes $b$ in a very similar manner to $K_{ds}$, whereas $K_2$ vastly overestimates, particularly for large $b$. The significance here is that one can calculate the residue momentum distribution of $\bar{K}_2$, but not $\bar{K}_3$, so the reduction of the cancellation is critical to obtaining a reliable estimate for the diffractive-stripping residue momentum distribution.

It is also constructive to consider $d\sigma/ds_d$, where $s_d$ is the projection of $\vec{r}$ onto the impact parameter plane for the diffracted nucleon, calculated only for the first part $K_2$. The original arrangement using $|S_i|^2$ gives the largest contributions when the nucleon is furthest from the target so we would expect larger contributions from the interior of the nucleon wave function (smaller $s_d$). The calculations for $d\sigma/ds_d$ are shown in Fig. 6.7. When using the original arrangement of the scattering operator, the first term $K_2$ probes the interior of the wave function, but such contributions will be removed by the projection-off term. It is also suggested that the diffraction mechanism probes the peripheral components of the nucleon wave function moderately more than stripping, though how $\bar{K}_3$ probes $s$ is unclear. Fundamentally, residue momentum calculations based on $K_2$ will be inaccurate since the interior of the wave function is probed far too strongly. Based on the estimates of $R_{\lambda\lambda}^j$, we see that the intrinsic momentum distribution becomes wider for small $s$, so we might expect full calculations based on $K_2$ to give distributions broader than expected.

We then compare the heavy residue momentum distributions resulting from the two arrangements of the operator in Fig. 6.8. Using the rearranged form the correction of $\bar{K}_3$ is just $15\%$, so the heavy residue momentum distribution arising from $\bar{K}_2$ is expected to be
Figure 6.6: Stripping-diffraction cross section as a function of core impact parameter for different arrangements of the diffraction operator; $K_2$ (dotted), $\tilde{K}_2$ (dashed), $\tilde{K}_2 - \tilde{K}_3 = K_2 - K_3 = K_{ds}$ (solid) and stripping (open circles). Altering the diffractive operator $K_2 \rightarrow \tilde{K}_2$ vastly reduces the cancellation effect of $K_3$ and constrains the cross section arising from the first part of the diffraction operator to a much smaller range of $b$. The total ($K_{ds}$, solid line) is unaltered by the substitution.
Figure 6.7: Cross section for two-proton removal from $^{28}$Mg as a function of $s$ for the diffracted nucleon for $\mathcal{K}_2$ (dashed line) and $\tilde{\mathcal{K}}_2$ (solid line), where the former has been normalised to the same area latter. Also shown is corresponding stripping cross section (open circles). The diffraction cross section is marginally more peripheral than the stripping, though the localisation of the correction from $\tilde{\mathcal{K}}_3$ is not clear. The adjusted operator $\tilde{\mathcal{K}}_2$ describes the expected surface localisation better than $\mathcal{K}_2$. 
representative of the physical $d\sigma_{ds}/d\kappa_c$. In this case the result is matches pure-stripping
events very well, as expected from the similar surface localisation of the reaction mecha-
nisms. The original arrangement is very different to the pure stripping case, particularly
for $I = 0$. The momentum-integrated cross sections related to $\bar{K}_3$ can be calculated as
described in Ref. [9] and Section 2.2.6, to obtain the momentum integrated cross section.

6.5 Centre of mass impact parameter

We investigate the sensitivity of the residue momentum distributions to the peripherality
of the reaction, discussed here in terms of the minimum centre of mass impact parameter
$b$. Widths of the residue momentum distributions following one-nucleon knockout are
expected to be sensitive to the projectile impact parameter (see [5, 61]). Eikonal one-
nucleon knockout and Coupled Discretized Continuum Channels (CDCC) calculations
were compared for a range of residue angles (relative to the incident beam), with the
calculations essentially coincident for the smallest angles [68]. If the residue momentum
distribution were to be strongly sensitive to the centre of mass impact parameter it might
offer a further test of the reaction mechanism. It is expected that for the present cases,
where the nucleon binding energy is large, that cross section will fall rapidly as $b$ increases.

Here we assume that the most forward going residues will result from the largest
projectile centre of mass impact parameters, where there will be the weakest interaction
of the core and target. We investigate the sensitivity of the residue momentum distribution
to the centre of mass impact parameter by setting the minimum value for the integral
over $b$. The values of $b$ are chosen so as to give a certain fraction of the total stripping
cross section, such that if one were to take the most forward-going residues there may
be some effect on the residue momentum distribution. The calculations for $I = 0$ and
$I = 4$ are shown in Fig. 6.9. The $I = 4$ case narrows significantly for more peripheral
reactions, whereas the $I = 0$ case does not. The sensitivity is however relatively weak
and the cross section falls very rapidly, such that a high precision experiment would be
required to investigate such an effect.
Figure 6.8: Diffractive-stripping residue momentum distributions for $\pi[0d_{5/2}]^2$ removal from $^{28}\text{Mg}$ showing contributions from the first part of the original diffractive term $\mathcal{K}_2$ (dotted), the first part of the rearranged diffractive term $\bar{\mathcal{K}}_2$ (dashed) and stripping term $\mathcal{K}_1$ (solid). The top panel shows $I = 0$ and the lower panel shows $I = 4$. The shape of the residue momentum distribution is clearly incorrect for the original arrangement in the $I = 0$ case and the large overestimation in both cases is apparent.
Figure 6.9: Sensitivity to the minimum centre of mass impact parameter on the residue longitudinal momentum distribution shape for pure $\pi[0d_{5/2}]^2$ knockout from $^{28}\text{Mg}$ for $I = 0$ (solid lines) and $I = 4$ (dashed lines). The minimum allowed values of the projectile centre of mass impact parameter $b$ are, for $I = 0$, 0.00, 6.72 and 7.57 fm and, for $I = 4$, 0.00, 6.79 and 7.64 fm respectively. These values of $b$ correspond to a certain fraction of the cross section, being 100%, the outermost 10% and outermost 1% respectively. The $I = 4$ case narrows much more quickly than the $I = 0$ case, but the difference is relatively small given the very large reduction in the cross section. All curves have been normalised to the same peak.
6.6 Total orbital angular momentum and nucleon pair configuration

We saw in Section 4.2 that the shape of the heavy residue momentum distribution is given by the incoherent sum of different couplings of the total orbital angular momentum of the two nucleons $L$. The assumption has been that larger $L$ will lead to broader residue momentum distributions. Whilst the width may be fundamentally determined by $L$, it is worth remembering that even for an even-even projectile and a single two-nucleon configuration several $L$ values may contribute to a given final state. This will be further complicated when several configurations contribute, each of which will favour different values of $L$, which will not have identical distribution even for a particular $L$. Further, there will typically be cross terms ($\alpha \neq \alpha'$) that we do not consider here. Nevertheless, some configurations will allow only a single $L$. Where one of the orbitals is $s$-wave, the $L$ value will simply be equal to the orbital angular momentum of the second nucleon and some states following unlike pair (np) knockout will also be constrained to a particular $L$.

Essentially we evaluate the last two lines of Eq. 2.106 (i.e. Eq. 2.109) to obtain $d\sigma_{\text{str}}^{\alpha \alpha' L}/d\kappa_c$. Combinations of these must be added incoherently to give the distribution for a particular $I$. We first consider the distributions for different $L$ for $\pi[0d_{3/2}]^2$ knockout from $^{28}\text{Mg}$, shown in Fig. 6.10. The residue momentum distribution width increases as the total orbital angular momentum $L$ increases, so configurations coupled to a particular $\vec{L} + \vec{S} = \vec{I}$ that emphasize larger $L$ will have broader distributions. The difference in width is sufficiently large that subtle differences in the weighting of $L$ might give clearly different residue momentum distributions, particularly for large $I$. Further, the distributions get wider more quickly as $L$ increases.

Consideration of $LS$ coupling gives additional insight into the influence of the pair combination on the residue momentum distribution. The relative strengths of different $L$ for a particular two-nucleon configuration are determined by the two $9j$ coefficients and by statistical factors (see first part of Eq. 4.14). We denote this by $W_{\alpha \alpha' L S I}^{LSI}$,

$$W_{\alpha \alpha' L S I}^{LSI} = \frac{\hat{\ell}_1 \ell_1 \hat{\ell}_2 \ell_2 \hat{j}_1 j_1 \hat{j}_2 j_2}{(4\pi)^2} \left\{ \begin{array}{ccc} \ell_1 & s & j_1 \\ \ell_2 & s & j_2 \\ L & S & I \end{array} \right\} \left\{ \begin{array}{ccc} \ell_1' & s & j_1' \\ \ell_2' & s & j_2' \\ L & S & I \end{array} \right\} \hat{L}^2 \hat{S}^2. \quad (6.1)$$

The sign of the weight is determined by the two Racah coefficients. Where there is a single configuration the two Racah coefficients are identical and the weight is positive, but if two different configurations are involved ($\alpha \neq \alpha'$) then the weight can be negative.

As an example, we consider two-nucleon removal from the $0d_{3/2}$ and $0d_{5/2}$ orbitals,
6.6. TOTAL ORBITAL ANGULAR MOMENTUM

Figure 6.10: Residue momentum distributions (see Eq. 2.109 for \(L = 0\) (solid), 1 (dotted), 2 (dashed), 3 (open circles) and 4 (open squares) for \(\pi [0d_{5/2}]^2\) removal from \(^{28}\text{Mg}\), which are independent of \(I\). All curves are normalised to the same peak. Note that the width increases more as \(L\) increases, consistent with the fact that the calculated \(I = 2\) distribution is more similar to \(I = 0\) than \(I = 4\) (see Fig. 6.1).

showing the weights of different \(L\) for each \(I\), as given by Eq. 6.1, in Table 6.2. We also show the isospin \(T\) required for the total spin-space-isospin wave function to be antisymmetric. Consideration of like-nucleon knockout will involve \(T = 1\) components only.

We first consider the \(\pi [0d_{5/2}]^2\) configuration, and note that the fully calculated residue momentum distributions shown in Fig. 6.1 indicate that the \(I = 2\) coupling case is more similar to the \(I = 0\) distribution than the \(I = 4\), i.e. there is a larger increase in width going from \(I = 2 \rightarrow 4\) than \(I = 0 \rightarrow 2\). The \(LS\) coupling weights in Table 6.2 show that for this configuration the dominant \(L\) values are for \(I = 0, L = 0\) and 1, for \(I = 2, L = 1\), and for \(I = 4, L = 3\). Since the dominant \(L\) in the \(I = 2\) case is closer to those of the \(L = 0\) case, we might expect these residue momentum distribution to be more similar.

We now compare the single shell \([0d_{5/2}]^2\) removal to that from different shells, namely \([0d_{5/2}][0d_{3/2}]\), for \(I = 4\). The weights for these cases are again shown in Table 6.2. The single shell combination weights \(L = 3\) most strongly, whereas the mixed shell knockout \((T = 1)\) is predominately \(L = 4\). We would thus expect the residue momentum distribution to be somewhat broader for the latter case than for the former. This is verified by the full distributions, shown in Fig. 6.11. The \([0d_{5/2}][0d_{3/2}]\) configuration gives a distribu-
### Table 6.2: Weights for the First Direct Term

The table below shows the weights for the first direct term. The relative weights of different $L$ and $S$ for a particular coupling $I$ depend on the configuration. The isospin $T$ is also shown and is that required to give an overall antisymmetric spin-space-isospin wave function. This simple calculation of two Racah coefficients and a statistical factor gives an indication of differences in residue momentum distributions for particular nucleon pairs. The sign of the weight is determined by the Racah coefficients and for the present cases (single configuration $\alpha$ and direct term only) are all positive.

<table>
<thead>
<tr>
<th>$[0d_{5/2}]^2$</th>
<th>$[0d_{5/2}] [0d_{3/2}]$</th>
<th>$[0d_{3/2}]^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$ $L$ $S$ $T$</td>
<td>$W_{\alpha\alpha'}^L$</td>
<td>$I$ $L$ $S$ $T$</td>
</tr>
<tr>
<td>0 0 0 1</td>
<td>0.095</td>
<td>0 0 0 1</td>
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<td>1 1 1 0</td>
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<td>1 1 1 0</td>
</tr>
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</tr>
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</tr>
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<td>2 1 1 1</td>
</tr>
<tr>
<td>2 0 1 0</td>
<td>0.076</td>
<td>2 0 1 1</td>
</tr>
<tr>
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<td>0.011</td>
<td>2 1 0 0</td>
</tr>
<tr>
<td>3 2 1 0</td>
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<td>3 1 0 0</td>
</tr>
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</tr>
<tr>
<td>5 4 1 0</td>
<td>0.158</td>
<td>5 4 1 0</td>
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</tbody>
</table>
6.6. TOTAL ORBITAL ANGULAR MOMENTUM

Figure 6.11: Examples of $I = 4$ heavy residue momentum distributions for $^{28}\text{Mg}(-2p)$ with different nucleon pair combinations. Solid lines and open points show $[0d_{5/2}]^2$ removal, dashed lines and solid points show $[0d_{5/2}][0d_{3/2}]$ removal. The full distributions are normalised to the same value at $\kappa_c = 0$, with the $L$ partial distributions scaled by the same factor. Circles show $L = 3$ contributions and squares show $L = 4$ with the total shown by the solid line. The different relative strengths of $L = 3$ and 4 for the different nucleon pair configuration gives a significantly different heavy residue momentum distribution for the same $I$ (dashed and solid curves).

...tion distinctly broader than from $[0d_{5/2}]^2$. Simply by expressing the residue momentum distributions in $LS$ coupling we have gained the ability to predict subtle differences in the residue momentum distributions associated with the (previously rather opaque) pair combination. Further, high precision, final-state exclusive residue momentum distributions may be able to distinguish the relative strengths of the different two-nucleon amplitudes.

Similarly, we can compare the $I = 2$ states populated by the $[0d_{5/2}]^2$ and $[0d_{3/2}]^2$ configurations. The relative strengths of the different $L$ values would suggest that the former combination, predominantly probing $L = 1$ and 2, should be significantly narrower than the latter combination, probing predominantly $L = 3$. This is indeed the case as is shown in Fig. 6.12. Once again this shows the usefulness of $LS$ coupling for understanding differences in the residue momentum distribution associated with the nucleon pair combination. The subtle differences in the component $L$ distributions for the different configurations can be attributed to differing radial wave functions.
Figure 6.12: Residue momentum distributions for $I = 2$ for the case of $^{28}\text{Mg}(−2p)$. The lines are for the configurations $[0d_5/2]^2$ (solid line, open symbols) and $[0d_3/2]^2$(dashed line, solid symbols). The lines (no symbols) show the total distributions for the two configurations and it is clear that the $[0d_3/2]^2$ configuration (dashed) gives a significantly broader distribution than the $[0d_5/2]^2$ configuration, as expected. The symbols show the contributions from $L = 1$ (circles), 2 (squares) and 3 (triangles). The full distributions are normalised to the same peak value, with the $L$ partial distributions scaled by the same factor. The minor differences in the residue momentum distributions for the same $L$ but different configurations (see e.g. $L = 2$, squares) can be attributed to differences in the radial wave function for the different configurations.
A further interesting example is removal of a pair of nucleons with configuration \( [s_{1/2}] [ℓ_j] \). In this case we can populate states with \( I = j - 1/2, j + 1/2 \), and since the first orbital is \( s \)-wave we are constrained such that \( L = ℓ \) and both the \( I = j - 1/2 \) and \( j + 1/2 \) states will have identical residue momentum distributions.

Further, consider population of two states following two-nucleon removal. In one case, the nucleon pair configurations is \( [s_{1/2}] [ℓ_{ℓ+1/2}] \) and the other \( [s_{1/2}] [(ℓ + 2) ℓ_{ℓ+2-1/2}] \). Both combinations can populate states of spin \( I = ℓ + 1 \), but the former will be pure \( L = ℓ \) and the latter pure \( L = ℓ + 2 \), and as such, the latter state should have a much wider residue momentum distribution than the former, despite the final state spins being the same. An example is provided by \( ^{208}\text{Pb}(−2p) \), where two \( 3^+ \) states are populated in \( ^{206}\text{Hg} \). One of which has the dominant configuration \( π[2s_{1/2}]^{−1}[1d_{5/2}]^{−1} \) and the other \( π[2s_{1/2}]^{−1}[0g_{7/2}]^{−1} \). The former will be pure \( L = 2 \) and the latter will be pure \( L = 4 \), giving the two states very different heavy residue momentum distributions, despite the final states having the same spin-parity \( J_f^π = 3^+ \). This particular case is discussed in Section 7.2.3.

It should be noted, the physical examples will be more complicated than the simple limits discussed here. Usually several configurations contribute to population a particular final state and, although sometimes a single configuration will dominate, many of the most interesting cases are rather mixed. We have not considered cases where two configurations are not the same (i.e. \( α ≠ α' \)). Expression of the problem in \( LS \) coupling does however provide some insight into the effects of the nucleon pair configuration.
Chapter 7

Applications

Here we consider several applications of two-nucleon knockout using the formalism developed in Chapter 2. This work has been published in peer-reviewed journals and these papers can be found in Appendix B. The first set of examples consider residue momentum distributions for two-nucleon knockout reactions in the \textit{sd}−shell: $^{22}\text{Mg}(−2n)$ [72], $^{38}\text{Si}(−2p)$ [12] and $^{28}\text{Mg}(−2p)$ [7]. Excellent agreement with experimentally measured residue momentum distributions is found. We then go on to discuss two-nucleon knockout from heavy systems, taking two-proton knockout from $^{208}\text{Pb}$ as our example, exploited in recent isomer-decay spectroscopy experiments [18, 73]. The primary complication for heavy mass projectiles is the high density of residue final states below particle separation thresholds, which, in the absence of prompt $\gamma$-decay measurements, makes estimation of isomeric population ratios difficult.

In addition to the examples discussed here, a systematic study was made of two-neutron removal reactions for neutron-rich carbon isotopes. The removal of two weakly bound nucleons requires careful consideration of indirect population of the final residue states via single-nucleon knockout to particle-unbound states. We do not discuss this example in detail here; it forms the basis of Ref. [31], which can be found in Appendix B.1. The primary conclusion is that the indirect process dominates the two-nucleon removal cross section for weakly bound nucleons.

7.1 Residue momentum distributions for systems in the \textit{sd}-shell

Two-nucleon knockout has been applied most extensively to exotic \textit{sd}-shell nuclei. In its most simple form, the reaction is used as a tool to populate excited states in very exotic species in order to perform $\gamma$-decay spectroscopy, e.g. $^{34}\text{Mg}(−2p)$ [38] and $^{22}\text{Mg}(−2n)$ [72].
Other experiments measure final-state exclusive cross sections allowing one to deduce the suppression of shell-model strength [9, 74] or investigate the evolution of shell structure [72, 75, 13]. Recent examples have focussed on nuclei near the island of inversion [10] and have deduced large changes in structure from particularly small two-nucleon removal cross sections. The island of inversion defines a region of nuclei near $^{31}$Na whose ground states are dominated by deformation-driving $2\hbar\omega$ (two-particle-two-hole) components due to a tensor-force reduced $sd - fp$ shell gap. The sudden change in structure reduces the initial-final state overlap, giving a small cross section. A similar island of inversion is predicted to exist at heavier masses, centred about $^{64}$Cr, which again has been studied via two-nucleon knockout [15].

In all such experiments measurement of residue longitudinal momentum distributions would give more detailed final-state information and offer verification of the reaction mechanism. The final state spins are often inferred from systematics, excited state energies or shell model calculations, but final state exclusive residue momentum distributions allow firm spin assignments to be made. Here we discuss three experimental examples where the residue momentum distributions have been measured, namely $^{22}$Mg($-2n$), $^{38}$Si($-2p$) and $^{28}$Mg($-2p$). In the $^{28}$Mg($-2p$), the high secondary beam intensity permitted the extraction of final-state exclusive residue momentum distributions for the first time. These examples were chosen primarily because of the quality of the experimental data available. Other examples are presented in Refs. [7] and [40], but are of lower statistically quality and suffer from strong experimental broadening. The results from this Section were published in Refs. [19] and [20] and these publications are reproduced in full in Appendices B.2 and B.3.

In the following three examples, the required $S$-matrices were calculated as described in Section 2.1.3 at the appropriate mid-target projectile energy. The projectile and residue radial density profiles were taken from Hartree-Fock calculations [43] using the SkX effective interaction [76]. The real-to-imaginary forward scattering amplitudes were taken from the table of Ref. [36], appropriate for the lowest energy tabulated, 100 MeV/nucleon. A Gaussian finite range nucleon-nucleon interaction was used, with range parameter $\beta = 0.5$ fm. These are essentially identical to those used in the previously published work on these examples (i.e. Refs. [9], [12] and [72]).

### 7.1.1 Two-neutron knockout from $^{22}$Mg

The main sensitivity of the residue momentum distribution was shown in Chapter 6 to be the total angular momentum $I$ of the two nucleons, with larger couplings leading to wider residue momentum distributions. The ideal test case would be final-state exclusive distributions resulting from two-neutron removal from a neutron-deficient system, since
7.1. EXAMPLES FROM THE SD–SHELL

Figure 7.1: Separation energy schematic for $^{22}\text{Mg}(−2\text{n})$, with separation energies taken from [48]. Since the removed neutrons are significantly more bound than the protons in $^{22}\text{Mg}$, the asymmetric nucleon separation energies ensure that the final residue states are populated directly (thick arrow) rather than indirectly (dashed arrows).

differential energy loss in the reaction target would be minimal. Here we discuss two-neutron removal from $^{22}\text{Mg}$ populating the $0^+$ ground and $2^+$ excited states in the $^{20}\text{Mg}$ residue.

The removal of two neutrons from neutron-deficient $^{22}\text{Mg}$ on a 188(4) mg/cm$^2$ beryllium target at 75 MeV/nucleon was studied experimentally in Ref. [72]. The primary motivation of the experiment was measurement of the first excited $2^+$ state for this $T=2$ isotope, and its place in the $T=2$ quintet of $^{20}\text{Mg}$, $^{20}\text{Na}$, $^{20}\text{Ne}$, $^{20}\text{F}$ and $^{20}\text{O}$. The two-nucleon knockout reaction was exploited to populate this state. A single $\gamma$-ray transition was observed, attributed to the decay of the first $2^+$ state (at 1.598 MeV) to the ground state. Though not discussed in Ref. [12], the final-state inclusive $^{20}\text{Mg}$ residue momentum distributions were measured in the experiment, and will be the primary interest here.

Since $^{22}\text{Mg}$ is neutron-deficient, we can be confident that the nucleon separation thresholds, shown in Fig. 7.1, will ensure that final states of the $^{20}\text{Mg}$ residue are populated directly. The proton separation threshold in $^{21}\text{Mg}$ $S_p = 3.226$ MeV is significantly lower than the neutron separation threshold $S_n = 14.730$ MeV, so any particle-unbound excited states in $^{21}\text{Mg}$ are expected to decay via proton-emission and so not populate states in $^{20}\text{Mg}$.

Of particular interest here is that the ground state is populated significantly more strongly than the excited $2^+$ state. The branching ratios, determined from $\gamma$-ray spec-

\[ S_n(^{22}\text{Mg}) = 19.379 + 3.226 + 14.730 \]
\[ S_{2n}(^{22}\text{Mg}) = 34.110 + 2.560 - 1n - 2n \]
\[ S_{p}(^{21}\text{Mg}) \]
\[ S_{p}(^{20}\text{Mg}) \]
troscopy, were 84% and 16% respectively [77]. Though only final-state inclusive momentum distributions were available, the strong ground state \((I = 0)\) branching gives a narrow experimental residue momentum distribution. The theoretical calculations overestimate the \(2^+\) branching ratio, giving 64% \(0^+\) and 36% \(2^+\), though whether this is due to theoretical underestimation of the ground state or overestimation of the \(2^+\) state is not clear in the absence of absolute cross sections. Assuming the ground state is correctly predicted the \(2^+\) state is then overestimated by a factor of three. This statement should be tempered by the absence of error bars for the experimental branching ratios.

Certainly the \(2^+\) state is expected to be weakly populated in this reaction and the small cross section is consistent with other examples in the \(sd\)-shell; the quantity \(\sigma(2^+)/[\sigma(0^+) + \sigma(2^+)]\) is in the range \(0.1 - 0.2\) for \(^{28}\text{Mg}(-2p), \text{\ }^{26}\text{Si}(-2n)\text{ and } \text{\ }^{34}\text{Ar}(-2n)\) [9]. As \(^{22}\text{Mg}\) is an \(N = 10\) nucleus the two removed neutrons are likely be the those in the \(sd\)-shell. The low energy excited states in \(^{20}\text{Mg}\) will be based primarily on excitations of the four protons outside a closed \(^{16}\text{O}\) core. The overlap of such excited states and the \(^{22}\text{Mg}\) ground state, where the four protons will be predominantly paired to angular momentum zero, is expected to be small. The alternative is dynamic excitation of the heavy residue in the reaction, which is neglected (spectator-core approximation).

The required two-nucleon amplitudes were taken from oxbash shell model calculations [43] using a full \(sd\)-model space and the \(USD\) effective interaction [78]. These are shown in Table 7.1 for the observed \(0^+\) and \(2^+\) states and the unobserved \(4^+\) state. The \(4^+\) state is predicted to lie \(\sim 1\) MeV above the proton separation energy \(S_p(^{20}\text{Mg})=2.645\) MeV [48]. Since only a single 1.589 MeV \(\gamma\)-ray transition is observed, the \(4^+\) state, if populated, it is presumed to decay via proton emission. Regardless of whether the state is proton-unbound, the shell model calculations suggest it would be extremely weakly populated. This is partly because it is accessible by fewer configurations (\([0d_{5/2}]^2\) and \([0d_{5/2}]\times[0d_{3/2}]\) only) and partly as coupling the neutron pair to \(I = 4\) in the \(^{22}\text{Mg}\) ground state is even more exotic than coupling them to \(I = 2\). Were we to consider two-neutron removal from \(^{24}\text{Mg}\), the strength to excited states should be much stronger since it is then possible to break two pairs of neutrons each coupled to \(I = 0\). This is borne out by the shell model calculations for this case, also shown in Table 7.1. The \(4^+\) state would be populated 10000 times more strongly in the \(^{24}\text{Mg}(-2n)\) case. Interestingly, the two-nucleon amplitudes are of the same magnitude for the first \(2^+\) state in both cases.

The core-nucleon wave functions were calculated in a Woods-Saxon plus spin-orbit potential as described in Section 3.1, the geometry of which was fitted to reproduce Hartree-Fock, \((SkX\) effective interaction [76]), root-mean-square radii and separation energies, and the depth adjusted to reproduce the experimental effective separation energy. The two-neutrons are deeply bound, with two-neutron separation energy \(S_{2n}(^{22}\text{Mg}) = 34.110\)
7.1. EXAMPLES FROM THE SD–SHELL

| $J^+_f$ | $|0d_{5/2}|^2$ | $|0d_{3/2}|^2$ | $|1s_{1/2}|^2$ | $|0d_{3/2}| |0d_{5/2}|$ | $|0d_{5/2}| |1s_{1/2}|$ | $|0d_{5/2}| |1s_{1/2}|$ |
|---------|--------------|--------------|--------------|-----------------|-----------------|-----------------|
| 0+      | 0.8029       | 0.2546       | 0.3784       | –               | –               | –               |
| 2+      | 0.4566       | 0.1010       | –            | -0.1937         | 0.5244          | -0.1983         |
| 4+      | -0.0153      | –            | –            | -0.0175         | –               | –               |
| 0+      | -1.2600      | -0.5187      | -0.3453      | –               | –               | –               |
| 2+      | 0.1516       | -0.0454      | –            | -0.6136         | -0.1212         | 0.0570          |
| 4+      | 1.6780       | –            | 0.7129       | –               | –               | –               |

Table 7.1: Two-nucleon spectroscopic amplitudes for the $^{22}\text{Mg}(0^+)^{\rightarrow}^{20}\text{Mg}(J^+_f)$ reaction calculated using the oxbash shell model code, with a full $sd$–model space and the USD interaction, as described in Ref. [72] (first three lines). The $4^+$ state predicted to have an excitation energy of 3.771 MeV and so is predicted to be proton-unbound. In any case it will be extremely weakly populated. The bottom section shows the two-nucleon amplitudes for $^{24}\text{Mg}(0^+)^{\rightarrow}^{22}\text{Mg}(J^+_f)$ for the ground and first $2^+$ and $4^+$ states. The magnitudes of the TNA are larger due to the increased number of valence neutrons, but the $4^+$ state in particular would be populated four orders of magnitude more strongly.

<table>
<thead>
<tr>
<th>$J^+_f$</th>
<th>Orbital</th>
<th>$R_{HF}\text{ (fm)}$</th>
<th>$B_{HF}\text{ (MeV)}$</th>
<th>$r_0\text{ (fm)}$</th>
<th>$B_{c_{exp}}^{0+}\text{ (MeV)}$</th>
<th>$R_{th}\text{ (fm)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0+</td>
<td>$0d_{5/2}$</td>
<td>3.238</td>
<td>13.18</td>
<td>1.320</td>
<td>17.06</td>
<td>3.204</td>
</tr>
<tr>
<td></td>
<td>$0d_{3/2}$</td>
<td>3.502</td>
<td>6.15</td>
<td>1.359</td>
<td>17.06</td>
<td>3.160</td>
</tr>
<tr>
<td></td>
<td>$1s_{1/2}$</td>
<td>3.350</td>
<td>10.00</td>
<td>1.157</td>
<td>17.06</td>
<td>3.086</td>
</tr>
<tr>
<td>2+</td>
<td>$0d_{5/2}$</td>
<td>3.238</td>
<td>13.18</td>
<td>1.320</td>
<td>17.85</td>
<td>3.183</td>
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<tr>
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<td>$0d_{3/2}$</td>
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<td>6.15</td>
<td>1.359</td>
<td>17.85</td>
<td>3.141</td>
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<tr>
<td></td>
<td>$1s_{1/2}$</td>
<td>3.350</td>
<td>10.00</td>
<td>1.157</td>
<td>17.85</td>
<td>3.059</td>
</tr>
</tbody>
</table>

Table 7.2: Hartree-Fock and calculated root-mean-square radii and binding energies, and fitted Woods-Saxon potential geometries for the $^{22}\text{Mg}$ valence neutron wave functions. The Hartree-Fock calculations use the $SkX$ effective interaction. A diffuseness parameter $a = 0.7$ fm was used and a spin-orbitl potential depth of $V_{so} = 6$ MeV.

We now discuss the $^{20}\text{Mg}$ residue longitudinal momentum distributions. The incident $^{22}\text{Mg}$ beam has a narrow momentum spread of 0.5%, corresponding to a momentum width of $\Delta K_{A+2} \approx 40$ MeV/c, which is essentially the only experimental broadening. As the two-neutron removal reaction does not change the charge of the projectile, the differential energy loss broadening in the thin 188(4) mg/cm$^2$ beryllium target is minimal and the experimental distribution, dominated by the $0^+$ ground state transition, is very narrow. The calculated projectile-rest-frame distributions are stretched by the Lorentz factor $\gamma$, convolved with the beam profile and centred on the experimental peak momentum. The contributions from the ground and $2^+$ excited states are included, scaled to the experimen-
Figure 7.2: $^{20}\text{Mg}$ residue momentum distributions following two-neutron knockout from $^{22}\text{Mg}$. The experimental data from Ref. [77] and the theoretical calculations for the ground state (dashed), $2^+$ state (dot-dashed) and total (solid) are shown, scaled to the experimental branching ratios. Using the theoretical branching ratios gives the dotted curve, which overestimates the experimental data on the high momentum side. The dot-dot-dashed curve shows the distribution for the $2^+$ final state scaled to the peak height, to emphasize the narrowness of the experimental data. The lowest momentum data point is most likely affected by the spectrometer acceptance. The experimental distribution is very narrow - just $\approx 300$ MeV/c FWHM - due to the intrinsic narrowness of the $0^+$ distribution and the absence of experimental broadening (c.f. the $^{38}\text{Si}(-2p)$ case, Fig 7.4).

Though the difference between the experimental and theoretical branching is reasonably large ($b_{\text{exp}}(0^+) = 84\%$, $b_{\text{th}}(0^+) = 64\%$), the resulting momentum distributions are similar (solid and dotted curves of Fig. 7.2). The largest difference occurs, as one might expect, at the extremes of the residue momentum. Even in this simple case of two final states with little experimental broadening, it is not clear that one could use a final-state inclusive residue momentum distribution to deduce the branching ratios of the final states.
with great precision. One could spot gross differences; whether a strongly populated state was bound or not, for instance.

Nevertheless, the agreement with present data is excellent, and though final-state inclusive, the reaction is dominated by the ground state transition giving a very narrow distribution. It is not possible to reproduce a narrow distribution such as this using an uncorrelated theory that is essentially the convolution of two single-nucleon removal distributions (see Section 6.1).

7.1.2 Two-proton knockout from $^{38}$Si

Two-nucleon knockout allows the study of the evolution of shell structure far from the valley of stability. Of particular recent interest is the island of inversion, which defines a region of neutron-rich nuclei around $^{31}$Na dominated by deformation-driving $2\hbar\omega$ intruder components due to the tensor force reduced $sd-fp$ neutron shell gap. The extent of the island of inversion and nature of the isotopes nearby, where the structure may change rather rapidly, is of interest. The two-nucleon removal cross section depends strongly on the size of the two-nucleon amplitudes, which express the parentage of the final residue in the projectile. A small cross section corresponds to a small overlap of initial and final states and, by implication, a large change in structure. Further, by making some assumption about the projectile and residue structure we can infer the proportion of intruder components.

The extent of the island of inversion is largely unknown, prompting the two proton removal experiment $^{38}$Si($-2p$) to clarify the nature of $^{36}$Mg [12]. The $^{38}$Si ground state is well described by $0\hbar\omega$ neutron configurations. The $^{36}$Mg ground and excited $2^+$ state were expected to contain strong $2\hbar\omega$ intruder components. The notation $n\hbar\omega$ refers to the number of particle excitations across a major shell gap, in this case, neutrons across the $sd-pf$ shell gap. Only the $0\hbar\omega$ components of $^{36}$Mg are accessible from (purely $0\hbar\omega$) $^{38}$Si, so taking the ratio of the experimental cross section to a theoretical cross section based on the assumption of pure $0\hbar\omega$ $^{36}$Mg gives the proportion of $0\hbar\omega$ components in $^{36}$Mg. Note that though the reaction removes two protons, it is neutron intruder configurations that are important in reducing the initial-final state overlaps. This procedure was followed in Ref. [12], and good agreement for the proportion of two-neutron-two-hole components was found between the knockout analysis and Monte-Carlo shell model calculations.

The experiment in question used a 376(4) mg/cm$^2$ beryllium reaction target with mid-target projectile energy of 83 MeV/nucleon. A single $\gamma$-ray transition was observed using $\gamma$-ray detectors situated at the reaction target, corresponding to the 0.660(6) MeV first excited $2^+$ state. The fractional populations of the ground and first $2^+$ states are 58(6)% and 42(6)% respectively. $^{38}$Si is already rather proton-deficient and we would expect the
Figure 7.3: Nucleon separation threshold schematic for $^{38}\text{Si}(-2p)$ taken from Ref. [48], showing the direct residue population (thick solid line) and indirect (dashed line) paths. The asymmetric separation energies in $^{38}\text{Si}$, $^{37}\text{Na}$ and $^{36}\text{Mg}$ ensure that the removal of two protons can only populate $^{36}\text{Mg}$ residue states via a direct (one-step) reaction.

$^{36}\text{Mg}$ residue states to be populated directly due to the asymmetric nucleon separation thresholds (see Fig. 7.3).

The calculations use two nucleon amplitudes taken from OXBASH shell model calculations [43] using the $sd-pf$-model space truncated to allow only $0\hbar\omega$ excitations, with the $SDPF-M$ effective interaction [79]. These were used in Ref. [12] and are tabulated in Table 7.3. The theoretical cross sections are thus calculated under the assumption that both $^{38}\text{Si}$ and $^{36}\text{Mg}$ contain only pure $0\hbar\omega$ excitations, that is, no excitation across the $sd-pf$ shell gap. This allows the deduction of the proportion of $0\hbar\omega$ partitions in $^{36}\text{Mg}$. Much like the case of $^{22}\text{Mg}(-2n)$ the ground and first excited $2^+$ states are populated and the absence of any additional $\gamma$-ray transition indicates that any additional excited states of significant spectroscopic strength must lie above the $^{36}\text{Mg}$ neutron separation energy $S_n = 2.8$ MeV, estimated from systematics. The neutron radial wave functions were calculated in a Woods-Saxon (central plus spin-orbit) plus Coulomb potential as described in Section 3.1, with diffuseness $a = 0.7$ fm and spin-orbit depth $V_{so} = 6$ MeV. The geometry of the potential well was fit to Hartree-Fock calculations and the fitted geometry used to calculate the radial wave function at the experimental effective separation energy. The details of the potential parameters radial wave functions are shown in Table 7.4.

Of particular interest here is the $^{36}\text{Mg}$ residue momentum distribution. The residue
7.1. EXAMPLES FROM THE SD−SHELL

Table 7.3: The \( sd − pf \) two-nucleon spectroscopic amplitudes \( C_{\alpha}^{J_fJ_I} \) for \( ^{38}\text{Si}(0^+)^{→^{36}\text{Mg}(J_f^\pi)} \), calculated using the oxbash shell model code [43] with the \( SDPF−M \) interaction in isospin formalism, as described in Ref. [12]. These two-nucleon amplitudes are calculated in a \( 0\hbar\omega \) model space, which allows no excitations across the \( sd − pf \) major shell gap.

<table>
<thead>
<tr>
<th>( J_f^\pi )</th>
<th>Orbital</th>
<th>( R_{HF} ) (fm)</th>
<th>( B_{HF} ) (MeV)</th>
<th>( r_0 ) (fm)</th>
<th>( B_{exp}^{J_J} ) (MeV)</th>
<th>( R_{th} ) (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0^+</td>
<td>0d_{5/2}</td>
<td>3.476</td>
<td>20.83</td>
<td>1.345</td>
<td>20.05</td>
<td>3.540</td>
</tr>
<tr>
<td>0d_{3/2}</td>
<td>3.499</td>
<td>15.04</td>
<td>1.354</td>
<td>20.05</td>
<td>3.433</td>
<td></td>
</tr>
<tr>
<td>1s_{1/2}</td>
<td>3.368</td>
<td>17.60</td>
<td>1.294</td>
<td>20.05</td>
<td>3.513</td>
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<tr>
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<td>0d_{5/2}</td>
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<td>20.83</td>
<td>1.345</td>
<td>20.38</td>
<td>3.533</td>
</tr>
<tr>
<td>0d_{3/2}</td>
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<td>15.04</td>
<td>1.354</td>
<td>20.38</td>
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<tr>
<td>1s_{1/2}</td>
<td>3.368</td>
<td>17.60</td>
<td>1.293</td>
<td>20.38</td>
<td>3.343</td>
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</tbody>
</table>

Table 7.4: Hartree-Fock and fitted root mean square radii \( R \) and separation energies \( B \) and fitted potential size parameters \( r_0 \). A spin-orbit potential of depth \( V_{so} = 6 \) MeV was used and diffuseness parameter \( a = 0.7 \) fm.

momentum distribution is essentially a property of nucleons in the initial state, but clearly only those that can be removed to populate a particular final residue. Though the assumption of pure \( 0\hbar\omega \) configurations for \( ^{36}\text{Mg} \) leads to an overestimate the momentum-inclusive cross section, the reaction genuinely probes only \( 0\hbar\omega \) components since the \( ^{38}\text{Si} \) wave function is predominately \( 0\hbar\omega \), and the residue momentum distribution should be well described. Unlike the \( ^{22}\text{Mg}(−2n) \) case the experimental broadening is significant. The incident beam had a broad momentum resolution of 1.66%, corresponding to a momentum width of \( \Delta K_{A+2} = 230 \) MeV/c. The differential energy loss in the thick \( 375(4) \) mg/cm\(^2\) beryllium reaction target causes additional (square) broadening of 290 MeV/c. Furthermore, when compared to the \( ^{22}\text{Mg}(−2n) \) case, the \( 2^+ \) state is more strongly populated in this case.

The calculated momentum distributions are shown in Fig. 7.4. They are scaled vertically to match the experimental distribution, with the \( 0^+ \) and \( 2^+ \) state components weighted by 58% and 42%, in accord with the experimental branching ratios from \( \gamma \)-ray data. The theoretical distributions are strongly broadened, but the agreement with the experimental data is excellent. The open circles show the theoretical calculations boosted to the lab frame, but excluding the strong broadening caused by the target and beam resolution. This result is considerably narrower than the convolved calculations, and is
7.1. EXAMPLES FROM THE SD–SHELL

Figure 7.4: Final state inclusive $^{36}$Mg residue longitudinal momentum distribution following $^{38}$Si($-2p$). The dashed and dotted curves show the $0^+$ ground and $2^+$ excited states respectively (scaled using the experimental branching ratios) with the solid line showing the total. The data points are from [77]. The open circles show the sum of the $0^+$ and $2^+$ states boosted to the lab frame (Lorentz stretch) but excluding the broadening effects of the finite beam momentum width and differential energy loss in the target. The large degree of experimental broadening, due to differential energy loss in the reaction target and poor incident beam resolution, reduces the distinction between the $0^+$ and $2^+$ contributions (c.f. $^{22}$Mg($-2n$) Fig. 7.2 with horizontal axis covering the same width as the present Figure).

similar in width to the $^{22}$Mg case shown in Fig. 7.2.

7.1.3 Two-proton knockout from $^{28}$Mg

The removal of two-protons from $^{28}$Mg was shown to proceed via a direct reaction in Ref. [7], describing the final state inclusive $^{28}$Mg residue longitudinal momentum distribution with the convolution of two $\pi[0d_{5/2}]$ single-proton removal distributions. This is approximately equivalent to uncorrelated two-proton knockout distributions discussed in Sections 2.4 and 6.1. The inclusive cross section was also reasonably described within an uncorrelated knockout model. Later, the final-state exclusive cross sections were discussed in the context of a fully-correlated model taking into account stripping [8], and later diffractive-stripping [9] mechanisms. $^{26}$Ne is rather less exotic than either of the residues from previous examples ($^{20}$Mg and $^{36}$Mg), having a neutron separation energy of $S_n = 5.530$ MeV. As a result, four $^{26}$Ne residue final states were observed, being the
Figure 7.5: Separation energies relevant to two-proton knockout from $^{28}\text{Mg}$ [48]. The heavy residue states in $^{26}\text{Ne}$ must be populated directly (thick arrow), since the small proton separation energies (relative to the neutron separation energies) prevent the indirect process (dashed arrows).

$J_f^z = 0^+$ ground state, the first (2.02 MeV) and second (3.70 MeV) $2^+$ excited states, and the first $4^+$ state (3.50 MeV). The relatively large statistics for this experiment allows, for the first time, the extraction of final-state exclusive residue momentum distributions for the ground and $4^+$ states. Nevertheless, the direct population of the $^{26}\text{Ne}$ final residue states is assured due to the asymmetric nature of the nucleon separation energies, as illustrated in Fig. 7.5.

Structurally, a naïve view of $^{28}\text{Mg}$, having four protons in the $sd$-shell would place all four in the $0d_{5/2}$ shell. We might therefore expect that two-nucleon configurations including this orbital to be strongest. As in Ref. [8] the required two-nucleon amplitudes were calculated using the shell model code OXBASH [43] with a full $sd-$shell model space and the USD effective interaction [78], and are reproduced in Table 7.5. For all four residue final states the $[0d_{5/2}]^2$ configuration is strongest, though the first $2^+$ state contains strong admixtures of the $[0d_{5/2}0d_{3/2}]$ configuration. As described in Section 6.1 we would expect the widths of the residue momentum distribution to increase as the final state spin increases, but it is harder to say if there would be subtle differences between the $2^+$ states. The configurations for both states are rather mixed and whilst the second $2^+$ state is at higher energy $E_f$, and so the removed nucleons more strongly bound, the difference in the individual nucleon binding is small - just 0.84 MeV - relative to the total binding of $\sim 16$ MeV, so we would expect only a small change in width due to the additional binding (see
7.1. EXAMPLES FROM THE SD−SHELL

Table 7.5: Two nucleon amplitudes for states in $^{26}\text{Ne}$ populated via two-proton removal from $^{28}\text{Mg}$, calculated using the oxbash shell model code [43] in an sd-shell model space and using the USD interaction. For all four states the $[0d_{5/2}]^2$ configuration contributes most strongly. The $2^+_1$, $4^+_1$ and $2^+_2$ excited states have energies 2.02, 3.50 and 3.70 MeV respectively. Note the error in the header of Table II of Ref. [8], pointed out in Ref. [9].

\begin{tabular}{cccccccc}
\hline
$J^\pi_f$ & Orbital & $R_{HF}$ (fm) & $B_{HF}$ (MeV) & $r_0$ (fm) & $B_{exp}^f$ (MeV) & $R_{orb}$ (fm) \\
\hline
0$^+$  & 0d$_{5/2}$ & 3.328 & 12.98 & 1.298 & 15.02 & 3.335 \\
      & 0d$_{3/2}$ & 3.526 & 6.03  & 1.332 & 15.02 & 3.281 \\
      & 1s$_{1/2}$ & 3.399 & 9.73  & 1.211 & 15.02 & 3.235 \\
2$^+_1$ & 0d$_{5/2}$ & 3.328 & 12.98 & 1.298 & 16.03 & 3.309 \\
      & 0d$_{3/2}$ & 3.526 & 6.03  & 1.332 & 16.03 & 3.258 \\
      & 1s$_{1/2}$ & 3.399 & 9.73  & 1.211 & 16.03 & 3.202 \\
4$^+$  & 0d$_{5/2}$ & 3.328 & 12.98 & 1.298 & 16.77 & 3.291 \\
      & 0d$_{3/2}$ & 3.526 & 6.03  & 1.332 & 16.77 & 3.242 \\
      & 1s$_{1/2}$ & 3.399 & 9.73  & 1.211 & 16.77 & 3.179 \\
2$^+_2$ & 0d$_{5/2}$ & 3.328 & 12.98 & 1.298 & 16.87 & 3.289 \\
      & 0d$_{3/2}$ & 3.526 & 6.03  & 1.332 & 16.87 & 3.240 \\
      & 1s$_{1/2}$ & 3.399 & 9.73  & 1.211 & 16.87 & 3.176 \\
\hline
\end{tabular}

Table 7.6: Hartree-Fock and calculated potential and wave function parameters for $^{28}\text{Mg}$ single-nucleon wave functions calculated using a Woods-Saxon (central plus spin-orbit) plus Coulomb potential with diffuseness parameter $a = 0.7$ fm and spin orbit potential depth $V_{so} = 6$ MeV. A spin-orbit potential of depth $V_{so} = 6$ MeV was used.

\begin{tabular}{cccccc}
\hline
$J^\pi_f$ & Orbital & $R_{HF}$ (fm) & $B_{HF}$ (MeV) & $r_0$ (fm) & $B_{exp}^f$ (MeV) \\
\hline
0$^+$  & 0d$_{5/2}$ & 3.328 & 12.98 & 1.298 & 15.02 \\
      & 0d$_{3/2}$ & 3.526 & 6.03  & 1.332 & 15.02 \\
      & 1s$_{1/2}$ & 3.399 & 9.73  & 1.211 & 15.02 \\
2$^+_1$ & 0d$_{5/2}$ & 3.328 & 12.98 & 1.298 & 16.03 \\
      & 0d$_{3/2}$ & 3.526 & 6.03  & 1.332 & 16.03 \\
      & 1s$_{1/2}$ & 3.399 & 9.73  & 1.211 & 16.03 \\
4$^+$  & 0d$_{5/2}$ & 3.328 & 12.98 & 1.298 & 16.77 \\
      & 0d$_{3/2}$ & 3.526 & 6.03  & 1.332 & 16.77 \\
      & 1s$_{1/2}$ & 3.399 & 9.73  & 1.211 & 16.77 \\
2$^+_2$ & 0d$_{5/2}$ & 3.328 & 12.98 & 1.298 & 16.87 \\
      & 0d$_{3/2}$ & 3.526 & 6.03  & 1.332 & 16.87 \\
      & 1s$_{1/2}$ & 3.399 & 9.73  & 1.211 & 16.87 \\
\hline
\end{tabular}

The radial wave functions are calculated with a Woods-Saxon (central plus spin-orbit) plus Coulomb potential, as described in Section 3.1 with diffuseness parameter $a = 0.7$ fm and spin orbit potential depth $V_{so} = 6$ MeV. The Hartree-Fock and calculated root mean square radii and binding energies are detailed in Table 7.6. Due to the large initial binding the proton orbital root-mean-square radii vary little between the different final states.

The original publication Ref. [7] presented only final-state inclusive $^{26}\text{Ne}$ residue momentum distribution. Recently the inclusive distribution has been re-analyzed and re-binned, and exclusive longitudinal momentum distributions for the $J_f^\pi = 0^+$ ground state
and $4^+_1$ excited state extracted. The $^{26}$Ne $4^+$ distribution was obtained by gating on the $4^+$ to $2^+$ $\gamma$-ray transition, observed at 1.48 MeV, in coincidence with the $^{26}$Ne residues. The $^{26}$Ne ground state longitudinal momentum distribution was obtained by subtracting the distributions in coincidence with the two $\gamma$-ray transitions observed in the data set at 1.48 MeV and 2.02 MeV ($2^+ \rightarrow 0^+$) from the inclusive distribution, after taking into account the $\gamma$-ray photo-peak detection efficiencies at these two energies [80].

In order to compare with the experimental data, the theoretical calculation is first boosted to the lab frame, stretching the distribution by $\sim 9\%$. This distribution is the convolved with the Gaussian shaped incident $^{28}$Mg beam profile, of full-width-half-maximum 100 MeV/c. Finally, the theoretical distributions were convolved with a 240 MeV/c square distribution, resulting from differential broadening in the 375(4) mg/cm$^2$ beryllium target [77].

The calculated residue momentum distributions, both inclusive and final state exclusive, are shown in Fig. 7.6. The agreement with the data is very good, particularly in the inclusive case where the error bars are small. The $0^+$ and $4^+$ exclusive distributions, though being of lower statistical quality, are also well reproduced. The pronounced low-momentum tail visible for the inclusive distribution is commonly observed in single-nucleon removal experiments. The theoretical distributions are symmetric since the eikonal approximation is energy non-conserving [68]. Continuum discretized coupled channels calculations for the diffractive part of single-nucleon knockout are able to reproduce the observed tails to some extent, but further contributions arise from the stripping events (see Ref. [30]).

### 7.2 Heavy projectiles - two-proton knockout from $^{208}$Pb

The examples discussed thus far have been confined to $sd - pf$ shell nuclei, the region of the nuclear chart most widely studied with two-nucleon knockout reactions. Recent experiments with nuclei near $N \approx 28$, studying neutron-rich silicon, phosphorous and sulphur isotopes [13, 14], and near $N \approx 40$, investigating structural changes in nickel, iron and chromium isotopes [15], have extended the use of two-nucleon knockout to the $A = 40-70$ region. These examples have shown the validity of two-nucleon knockout beyond the $sd$-shell and highlighted the versatility of the reaction as a probe of changes in nuclear structure. New radioactive beam facilities will provide a vast array of exotic radioactive beams spanning the range of the nuclear chart offering new applications of knockout reactions. Of particular interest here is the application of nucleon knockout to
7.2. HEAVY PROJECTILES

Figure 7.6: Measured (symbols) and theoretical (curves) inclusive and exclusive longitudinal momentum distributions after two-proton knockout from $^{28}$Mg at 82 MeV/u. Panel (a) shows the measured inclusive distribution and the theoretical total (solid) constructed from the weighted $0^+$ (dashed), $2_1^+$ (dotted), $4^+$ (solid, small open circles) and $2_2^+$ (dot-dashed) distributions. Panel (b) shows the $^{26}$Ne$(0^+, \text{g.s.})$ distribution and panel (c) the $^{26}$Ne$(4^+, 3.50 \text{ MeV})$ final state distribution. In both (b) and (c) the solid line shows the experimentally broadened (incident beam plus target) distribution and the dashed line shows the raw theoretical calculation after transforming to the lab frame. Panel (d) shows all distributions on a logarithmic scale. All the calculated shapes take account of the beam-profile (Gaussian, FWHM 0.1 GeV/c) and differential energy loss in the target (square, width 0.24 GeV/c) broadening.
heavily mass projectiles.

Studies on exotic nuclei by the Rare ISotope INvestigation at GSI (RISING) populate isomeric states in heavy, neutron-rich nuclei via knockout and fragmentation reactions, which subsequently decay via $\gamma$-emission, giving insight into the structure of very rare isotopes. Isomeric states have, for various reasons, inhibited decay, and are consequently long-lived. The population of isomers is essential to the technique, since any non-isomeric states decay promptly prior to the $\gamma$-ray detectors. This makes prediction of the fractional population of a particular isomeric state, the isomeric ratio $R$, of great value. Such experiments typically fragment $^{208}\text{Pb}$ or $^{238}\text{U}$ beams to produce a wide range of isomers. One application used isomeric states to study the angular momentum population following fragmentation of $^{208}\text{Pb}$ [81, 82], though it was found that the isomeric ratios were generally underestimated by conventional abrasion-ablation fragmentation models [83, 84]. Many of the isomers studied involved reactions where many nucleons were removed, but for the most peripheral cases (one or two nucleons) we should obtain precise isomeric ratios from a knockout model.

The primary complication in heavy mass cases is likely to be the high density of states below nucleon separation thresholds. Removing just two protons from $^{208}\text{Pb}$ populates 56 excited states in $^{206}\text{Hg}$, all of which are populated with reasonable spectroscopic strength. Removing four protons to populate $^{204}\text{Pt}$, a channel recently observed [18, 85, 86], gives several thousand excited states. Despite this complication such heavy-projectile cases are of interest as they provide an efficient method of producing very neutron-rich nuclei. Current experiments measure the decay of isomeric states only, but the richest spectroscopic information is obtained via measurements of prompt $\gamma$-decays and future experiments with heavy secondary beams will provide such measurements.

Here we consider the example of $^{208}\text{Pb}(-2p)$, populating isomeric states in $^{206}\text{Hg}$. The recent RISING campaign [18, 86] studied the fragmentation of $^{208}\text{Pb}$ on a thick 2.525 g/cm$^2$ 9Be target at 1 GeV/nucleon. The exotic fragments are created at the beryllium reaction target, identified using the Fragment Recoil Separator (FRS) and associated detectors, and then stopped using a perspex stopper, after which their $\gamma$-decay is observed using Germanium detectors. The flight time from reaction target to stopper is approximately 200 ns, during which any non-isomeric states decay. These prompt decays are not measured. This would necessitate placing Germanium detectors around the reaction target, which would be overwhelmed by X-rays and $\gamma$-rays from other reaction products. This additional complication, that the feeding of isomeric states via $\gamma$-cascades from higher lying states is essentially unknown, would be mitigated via measurements of prompt $\gamma$s, though this presents a considerable experimental challenge.
7.2. HEAVY PROJECTILES

7.2.1 Calculation details

Of interest here is the two-proton removal residue, namely $^{206}\text{Hg}$, for which two isomeric states are observed; the $5^-$ predominantly $\pi[0h_{11/2}][2s_{1/2}]$ hole state and $10^+$ pure $\pi[0h_{11/2}]^2$ hole state. In both cases the decay is inhibited simply due to the weakness of the $\gamma$-ray transition strengths. The $5^-$ decays via $E3$ emission to the first excited $2^+$ state, and the $10^+$ decays via either a low energy (100 keV) $E2$ to the first $8^+$ state or an $E3$ to the first $7^-$ state \[87\]. These isomers and the additional states observed in their $\gamma$-ray cascades are shown in Fig. 7.7. From the decay of the $10^+$ isomer it is clear that the $5^-$ isomer will be fed by at least the observed $7^-$ and $8^+$ states and the $10^+$ isomer. The sum of the direct population ratios for these states, offers a minimum value for the isomeric ratio of the $5^-$ state.

The projectile in the present case is less exotic than the $sd-$shell examples discussed in Section 7.1 and the nucleon separation thresholds of $^{208}\text{Pb}$ are more symmetric, as shown in Fig. 7.8. Nevertheless, the neutron separation threshold of $^{207}\text{Tl}$ is lower than the proton threshold (though only by 0.5 MeV), and states above the $^{207}\text{Tl}$ proton-threshold would be expected to decay via neutron emission. Further, we would expect the majority of the single-proton hole strength to be exhausted before the particle thresholds are reached; the (most deeply bound) $\pi[0g_{7/2}]$ proton-hole state in $^{207}\text{Tl}$ is at 3.48 MeV. In order to populate the $10^+$ isomer, states above 11.1 MeV in $^{207}\text{Tl}$ must be populated.

The required two-nucleon amplitudes were calculated using the OXBASH shell-model code \[43\] with the $jj56pn$ model space and $khhe$ interaction of Ref. \[88\]. The $jj56pn$ model space includes five proton orbitals ($2s_{1/2}$, $1d_{3/2}$, $1d_{5/2}$, $0g_{7/2}$ and $0h_{11/2}$). In principle, six neutron orbitals, but in the present case ($N = 126$) the neutron orbitals are completely filled. Note that since the the $khhe$ interaction is a hole interaction, the two-nucleon overlaps are calculated with reversed initial and final states, such that they must be multiplied by $\sqrt{2J_f + 1}$ \[89\]. The experimental excited state energies are well reproduced by the shell model calculations (see caption of Fig. 7.7).

The two-nucleon amplitudes for the $5^-$ and $10^+$ isomers, and the $7^-$ and $8^+$ state that feed into the $5^-$ isomer are shown in Table 7.7. The $8^+$ and $10^+$ states have pure $\pi[0h_{11/2}]^{-2}$ proton-hole configurations, as there are no other orbital pairs in the model space that can produce states of this spin-parity. The $5^-$ and $7^-$ states are predominantly based on the $[0h_{11/2}][2s_{1/2}]$ and $[0h_{11/2}][0d_{3/2}]$ configurations respectively, but small mixtures of other configurations also contribute.

In principle, the shell model can also be used to calculate transition matrix elements and thus transition strengths and $\gamma$-decay schemes. The model space does not allow
Figure 7.7: States observed experimentally via isomer decay spectroscopy in $^{206}$Hg following two-proton removal from $^{208}$Pb. The two isomeric states, being the short lived 10$^+$ and longer lived 5$^-$ are of interest here. These four states represent a small fraction of the total number of the populated states in this two-proton knockout reaction and are observed only because they are isomeric or are populated following the $\gamma$-decay of the isomers. The shell model calculation (see text) reproduces the experimental energies (2.102, 2.466, 2.623 and 3.723 MeV) well, giving 2.101, 2.361, 3.621 and 3.658 MeV for the 5$^-$, 7$^-$, 8$^+$ and 10$^+$ states respectively.
7.2. HEAVY PROJECTILES

Figure 7.8: The one- and two-nucleon thresholds for the $^{208}\text{Pb}$, $^{207}\text{Tl}$ and $^{206}\text{Hg}$ systems [48]. The direct (thick line, labelled $-2p$) and indirect (dashed lines, labelled $-1p$) two-proton removal paths to $^{206}\text{Hg}$ final states are indicated schematically.

| $J_f^\pi$ | $|0h_{11/2}\rangle$ | $|0g_{7/2}\rangle$ | $|0h_{11/2}\rangle$ | $|1d_{5/2}\rangle$ | $|0h_{11/2}\rangle$ | $|1d_{3/2}\rangle$ | $|0h_{11/2}\rangle$ | $|2s_{1/2}\rangle$ | $|0h_{11/2}\rangle^2$ |
|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $5^-\pi$  | 0.01431         | 0.13333         | 0.28217         | 0.94995         |                  |                 |                 |                 |                 |
| $7^-\pi$  | 0.03446         | 0.13024         | 0.99088         |                 |                  |                 |                 |                 |                 |
| $8^+\pi$  | -               | -               | -               | -               |                  |                 |                 | 1.0             |                 |
| $10^+\pi$ | -               | -               | -               | -               | -               |                 |                 | 1.0             |                 |

Table 7.7: Two-nucleon amplitudes for the high-spin states relevant to the isomeric states populated in $^{206}\text{Hg}$, calculated using OXBASH [43] using the $jj56\text{pm}$ model space and $kkhe$ effective interaction [88]. The configurations indicated atop the columns are proton-holes. The $8^+$ and $10^+$ states are pure $|0h_{11/2}\rangle$ two proton hole configurations. These TNA must be multiplied by $\sqrt{2J_f + 1}$. 
Table 7.8: Valence proton wave functions for the $^{208}$Pb($-2p$) case, calculated as described in Section 3.1. Shown are the Hartree-Fock ($HF$) and calculated ($th$) root mean square radii $R$ and binding energies $E$. The wave functions were calculated in Woods-Saxon (central plus spin-orbit) plus Coulomb potential. The diffuseness parameter was $a = 0.7$ fm and the spin-orbit potential depth $V_{so} = 6$ MeV.

<table>
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<th>Orbital</th>
<th>$R_{HF}$ (fm)</th>
<th>$E_{HF}$ (MeV)</th>
<th>$r_0$ (fm)</th>
<th>$E_{exp}$ (MeV)</th>
<th>$R_{th}$ (fm)</th>
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<td>5.421</td>
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<tr>
<td>$0h_{11/2}$</td>
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<td>9.16</td>
<td>1.276</td>
<td>9.36</td>
<td>6.121</td>
</tr>
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</table>

for $E1$ transitions [89]. Such transitions change parity between initial and final states and so the single-particle transition must involve the $0h_{11/2}$ orbital. It is not possible to couple this orbital to any other with angular momentum 1. However, some $E1$ strength is inferred by observed transitions in the analogous nucleus $^{130}$Sn [90]. There, the observed $8^+ \rightarrow 7^-$ transition lifetime cannot be explained without some $E1$ contribution, attributed to non-zero core admixtures. Since the shell model calculations would suggest few parity-changing transitions, the presence of $E1$ transitions might significantly alter the $\gamma$-decay scheme. Undue reliance on the shell-model decay-scheme is thus ill-advised.

The nucleon radial wave functions were calculated as described in Section 3.1, in a Woods-Saxon (central plus spin orbit) plus Coulomb potential, using a diffuseness parameter $a = 0.7$ fm and spin-orbit potential depth $V_{so} = 6$ MeV. The required Hartree-Fock calculations used the $SkX$ effective interaction [76]. Previously the nucleon binding energy has been taken to be half the two-nucleon separation energy plus the final residue state excitation. Due to the large number of final states, we have taken the nucleon binding energies to be the $^{208}$Pb single-proton separation energy plus the single-particle energy deduced from low-energy proton-hole states in $^{207}$Tl. Only five radial wave functions need then be calculated instead of nearly 300. In any case, the final-state excitations for the vast majority of states are unknown and would have to be taken from the shell-model. The same approach was used in Ref. [16]. The details of the radial wave functions are shown in Table 7.8.

The required $S$-matrices were calculated at the incident $^{208}$Pb projectile energy of 1000 MeV/nucleon, as described in Section 2.1.3. A zero-range nucleon-nucleon (NN) effective interaction was used with strength determined, in the usual way [36], from the free $nn$ and $np$ total cross sections. The real-to-imaginary ratios of the forward scattering nucleon-nucleon (NN) amplitudes were taken to be zero, since the NN amplitude is essentially
absorptive for the energy in question. The underlying (high-energy) approximations are expected to be particularly good at the energies considered in this example. The $^{206}$Hg density profile is taken from Hartree-Fock calculations using the $SkX$ interaction. The interaction was fit to, amongst many other data, the charge radius of $^{208}$Pb, and calculations were shown to accurately reproduce the charge-density profile of $^{208}$Pb [76]. In comparison to the cases discussed earlier, the incident projectile energy is more than an order of magnitude larger, in an energy regime where the eikonal approximation should be very good. As the projectile energy increases the diffractive-stripping mechanism becomes less important, being only $\sim$25% of the pure stripping cross section in the present case, compared to $\sim$75% in the case of $^{28}$Mg($-2p$) at 82 MeV/nucleon.

The thick 2.525 g/cm$^2$ target is used in order to optimise the experimental yield of more exotic isotopes studied in the experiment ($^{204}$Pt for example). Aside from broadening the resulting reaction residue momentum distributions, it is possible that two-step reactions will occur in the target. With regard to the present example, the $^{206}$Hg residue states could be populated by two sequential single-proton knockout events, populating $^{206}$Hg via the intermediate residue $^{207}$Tl. The probability of interaction in the target is small and the chance of two separate interactions even lower, but the single proton removal cross section considerably larger than that for two-proton removal. Using a simple method based on analytic integration through the thickness of the target (see Appendix A), we estimate that for the present case, the percentage of events arising from two-step process is $\sim$8% of the total, and so we neglect such events. However, precisely which final states such processes would populate is not clear; if they were to favour low-spin states for instance, the isomeric ratios could be (marginally) affected.

The calculated cross sections, including contributions from pure stripping, diffractive-stripping and pure diffraction (estimated), are shown in Table 7.9. All 56 states populated are shown, though the last four, based on strong $\pi[0g_{7/2}]^2$ proton-hole configurations are predicted to be neutron-unbound by the shell model. The cross sections is spread across all the states, with each contributing $\sim 1-4\%$ of the total strength. This is in stark contrast to the $sd$-cases where at most four states are populated and there is often considerable concentration of strength.

The calculated isomeric ratios are shown in Table 7.10. The $10^+$ isomer is assumed to be populated only directly, that is, without significant feeding from $\gamma$-cascades. It is the highest spin-state allowed by the model space, with proton-hole configuration $\pi[0h_{11/2}]^2$. The high-spin states above the isomer are expected to bypass the $10^+$ isomer; the $8^-$ (4.091 MeV) and $9^-$ and $8^-$ (5.331 and 5.729 MeV) states have predominant proton-hole configurations $\pi[0h_{11/2}] [2d_{5/2}]$ and $\pi[0h_{11/2}] [0g_{7/2}]$ respectively. When the $10^+$ isomer decays it cascades through $8^+$ and $7^-$ state and into the $5^-$ isomer. The $5^-$ isomer is
7.2. HEAVY PROJECTILES

Table 7.9: Calculated cross sections and population ratios for $^{208}$Pb($-2p$) for all 56 excited states. The total cross section $\sigma_{tot}$ is the sum of pure stripping, diffractive-stripping and (estimated) double diffraction components. The total cross section is 4.59 mb. Where experimental energies are available they are denoted by *. States that are above the $^{206}$Hg neutron-separation threshold $S_n = 6.729$ MeV, are denoted by †.

<table>
<thead>
<tr>
<th>$J^+_f$</th>
<th>$E_f$ (MeV)</th>
<th>$\sigma_{tot}$ (mb)</th>
<th>$R_{th}$ (%)</th>
<th>$J^+_f$</th>
<th>$E_f$ (MeV)</th>
<th>$\sigma_{tot}$ (mb)</th>
<th>$R_{th}$ (%)</th>
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7.2. HEAVY PROJECTILES

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</tbody>
</table>

Table 7.10: Theoretical and experimentally deduced [18] isomeric ratios for the $5^-$ and $10^+$ isomers in $^{206}$Hg. The errors on the $10^+$ state ratio include ambiguities associated with experimentally-missed $\gamma$-ray intensity of the 364 keV $7^- \rightarrow 5^-$ transition. Basing the isomeric ratio on the observed $10^+ \rightarrow 7^-$ and $8^+ \rightarrow 7^-$ transitions gives value (a). Additionally including the 364 keV transition gives (b). We also show the theoretical estimate for the $5^-$ state of Ref. [81], there based on the yields from the abrasion-ablation code ABRABLA [83].

thus fed by at least the $7^-$, $8^+$ and $10^+$ states, and it is possible that further unobserved transitions contribute to the observed yield.

Note that the calculated isomeric ratios assume that the spectrometer accepts the complete longitudinal momentum distribution. In reality, since the $^{206}$Hg residue is so close in mass and charge to the $^{208}$Pb beam, slits are placed in the spectrometer to eliminate the unreacted beam [91]. This will tend to cut reaction residues at the extremes of the momentum distribution, disproportionately cutting those residue final states with the broadest momentum distributions. Depending on i) the degree of the cut, ii) the width of the reaction induced momentum distribution relative to incident beam momentum width and iii) the significance of additional experimental broadening effects, the isomeric ratio could be significantly affected. We consider the consequences of experimental cuts on the residue momentum in the next Section.

7.2.2 Isomeric ratio as a function of residue momentum

The residue momentum distribution for the $10^+$ isomeric state is expected to be broad since the total angular momentum of the two nucleons is $I = 10$. It is interesting to consider the isomeric ratio as a function of the residue momentum, which could alter the observed isomeric ratio if only part of the momentum distribution is transmitted through the experimental apparatus. Naturally we would expect the isomeric ratio for this high-spin isomer to be significantly larger for large $\kappa_c$ than for the central momentum $\kappa_c = 0$. The $5^-$ isomer is predominantly fed by high-spin states, so we would again expect the
isomeric ratio to be larger at the extremes of the residue momentum. This effect can in principle be observed by gating on the inclusive residue momentum distribution, without the need to extract final-state exclusive residue momentum distributions.

We first consider the theoretical residue momentum distributions for the total cross section, and the $5^-$ and $10^+$ isomers. We also calculate the isomeric ratios as a function of the residue momentum $\kappa_c$. For the $5^-$ state we included contributions from the $7^-$, $8^+$ and $10^+(\text{isomeric})$ states. These calculations are shown in Fig. 7.9 for the projectile rest frame. The isomeric ratio depends sensitively on the residue momentum, with very large enhancements at the extremes. In the $5^-$ case it would be particularly interesting to measure the isomeric ratio at the central momentum, where we would expect it to be significantly smaller than the experimentally determined (momentum inclusive) value. If this were not the case (once experimental broadening had been taken into account) it would indicate additional feeding from low-spin states.

The effect of cutting the residue momentum distribution on the (momentum-integrated) isomeric ratio is not entirely apparent from Fig. 7.9. Though the isomeric ratio is much larger at the extremes of the momentum distribution, the cross section is strongly reduced, such that these regions will be less important in determining the total isomeric ratio. In Fig. 7.10 we show the isomeric ratio as a function of $\kappa_{\text{max}}$, where the isomeric ratio has been calculated assuming that only the events for which $-\kappa_{\text{max}} < \kappa_c < \kappa_{\text{max}}$ are transmitted through the spectrometer. This calculation tells us that in the absence of experimental broadening, the spectrometer must transmit $\pm 300\gamma$ MeV/c about the central momentum in order to correctly obtain the isomeric ratio, where $\gamma$ is the Lorentz factor.

In practice slits are placed within the fragment spectrometer which restrict the transmitted fragments on the basis of their $x$-position (that perpendicular to the beam direction). For the present case of the near-beam residue of $^{206}\text{Hg}$ the primary function of the slits is to eliminate the beam from the spectrometer to protect detectors downstream. The slits are placed after first bending magnet of the spectrometer, at which point the $x$-position is correlated with the fragment longitudinal momentum. The slits will have a tendency to remove the extremes of the momentum distribution, which will disproportionately affect residue final states which give broader residue momentum distributions i.e. high-spin states. Whether these cuts will affect the current experimental isomeric ratios depends on i) whether the slits are tight enough to significantly restrict the residue momentum, ii) the nature of such a cut i.e. whether it disproportionately cuts the extremes and where the peak position is relative to the center of the slits, and iii) whether experimental broadening effects preserve the distinctive widths of the different residue excited
Figure 7.9: $10^+$ (dashed) and $5^-$ ($7^-, 8^+, 10^+$, dotted) isomeric ratio as a function of heavy-residue momentum distribution. The top panel shows the residue momentum distributions themselves, with the $10^+$ and $5^-$ distributions scaled by a factor of $\times 20$ and $\times 5$ respectively to make comparison of the shape with the total (solid) easier. The middle and bottom panels show the isomeric ratio as function of the heavy residue momentum, with the experimental momentum-inclusive values shown by the horizontal line with the shaded area indicating the experimental error. In both cases the isomeric ratio is strongly enhanced at the edges of the distribution.
Figure 7.10: $10^+$ (dashed) and $5^-$ ($7^- , 8^+, 10^+$, dotted) isomeric ratios as a function of the maximum allowed momentum relative to the peak momentum, i.e. the isomeric ratio for the region $[-\kappa_{max} : \kappa_{max}]$. Both isomers show a strong dependence on the residue momentum cut such that experimentally determined isomeric ratios will only match the theoretical values if the spectrometer transmits the $\pm 300\gamma$ MeV/c about the central momentum, where $\gamma$ is the Lorentz factor.
states.

Note that the slit positions used in the following differ from those used in the published version Ref. [73], found in Appendix B.4. The slit positions used in the experiment are those of the published paper (0 – 10 mm), rather than those used here (±5 mm), but the following discussion offers an example of the potential reduction of isomeric ratios due to longitudinal momentum cuts. Those interested in the precise details of the current experiment are referred to Appendix B.4.

In the present experiment the spectrometer was tuned such that \( x = 0 \) corresponds to the peak position of \(^{208}\text{Hg}\), meaning that the \(^{206}\text{Hg}\) residue distribution is not centred in the middle of slits. The cut is thus asymmetric, cutting the low momentum side more severely. We first obtain the residue momentum distribution before the slits, excluding contributions from the reaction mechanism itself. Assuming the beam is mono-energetic, the \(^{206}\text{Hg}\) fragments will have a momentum distribution arising from differential energy loss in the 2.525 g/cm\(^2\) reaction target, since the location of the reaction vertex is unknown. This is calculated using LISE++ [92] with the conventional reaction-induced momentum distribution (a parametrized Gaussian) disabled. Broadening via other material prior to the first slit position is also included. The distribution of residues is given by LISE++ in terms of \( x \)-position at the entrance to the slits (±5 mm) and can be converted to momentum using

\[
\frac{\Delta p}{p} \approx \frac{\Delta x}{D},
\]

where the fractional change in momentum \( \Delta p/p \) is related to the change in \( x \)-position \( \Delta x \) divided by the dispersion constant \( D \) = 2150 mm [91], which is a parameter of the spectrometer. Knowing that \( x = 0 \) corresponds to \( p_0 = 308230.1 \text{ MeV/c} \) and \( \Delta p \ll p \approx p_0 \) allows us to use

\[
p = p_0 + \Delta p = p_0 \left( 1 + \frac{\Delta x}{D} \right).
\]

The projectiles lose energy in the passage through the target and the LISE++ calculations suggest the energy for \( x = 0 \) is 831.16 MeV/c. All cross sections are calculated at the incident projectile energy \( E = 1000 \text{ MeV/c} \). The cross section is expected to be weakly dependent on the projectile energy for the relativistic energies considered here, as the nucleon-nucleon cross section is weakly dependent on energy. The ratio of different final states, i.e. the isomeric ratios, are expected to be essentially independent of energy.

The Lorentz stretch is rather larger in this case than in the previous examples due to the much higher beam energy, stretching the distribution by a factor of 2.07. This transformation leaves the characteristic widths intact. In fact, the Lorentz boost dimin-
Figure 7.11: Theoretical cross sections as a function of laboratory frame residue momentum $K_A$, following convolution of the Lorentz-boosted projectile rest frame and the target-broadened $^{206}$Hg distributions. The $10^+$ (dashed line, $\times 20$), $5^-$ (dotted line, $\times 5$) and the inclusive (solid line) distributions are shown. The vertical lines show the momentum cuts imposed by the spectrometer slits; events between the lines are transmitted.

7.2.3 Example momentum distributions

Longitudinal residue momentum distributions have not been widely studied, even in the $sd$-shell. Experimental examples are thus rather limited in scope; the nucleons removed are restricted to $sd$-shell orbitals, and the maximum spin-state populated is $J^\pi = 4^+$. The $^{208}$Pb$(-2p)$ example offers more possibilities for interesting theoretical examples, as
### Table 7.11: Theoretical cross sections $\sigma_{str}$ and isomeric ratios $R_{th}$ for assuming transmission of the full residue momentum distribution, and cut residue momentum distributions, taking into account the effects of the slits. The last column shows the fraction of events stopped by the slits.

<table>
<thead>
<tr>
<th>State</th>
<th>$\sigma_{str}^{\text{full}}$ (mb)</th>
<th>$R_{th}^{\text{full}}$ (%)</th>
<th>$\sigma_{str}^{\text{cut}}$ (mb)</th>
<th>$R_{th}^{\text{cut}}$ (%)</th>
<th>$1 - \sigma_{cut}^{\text{cut}}/\sigma_{full}^{\text{full}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5−</td>
<td>0.683</td>
<td>18.7</td>
<td>0.504</td>
<td>18.1</td>
<td>0.25</td>
</tr>
<tr>
<td>10+</td>
<td>0.178</td>
<td>4.9</td>
<td>0.126</td>
<td>4.5</td>
<td>0.29</td>
</tr>
<tr>
<td>Total</td>
<td>3.65</td>
<td>100</td>
<td>2.79</td>
<td>100</td>
<td>0.24</td>
</tr>
</tbody>
</table>

i) there are a greater number of active orbitals covering a large range of orbital angular momenta, $\ell = 0−5$, and ii) higher spin states are expected to be populated in the reaction.

We first consider the states that can be populated via the $\pi[0h_{11/2}]^{-2}$ configuration, covering a large range of final-state spin, $J^\pi = 0^+−10^+$. We assume any final states are pure two-proton hole configurations, but in reality the physical states have mixed configurations, particular $J^\pi = 0^+$ and $2^+$. In Fig. 7.12 we compare the residue momentum distributions for the $\pi[0h_{11/2}]^{-2}$ pair for total angular momentum $I = 0−10$. It is clear that the increase in width with total angular momentum coupling is expected to persist for high-spin states. Also shown in Fig. 7.12 is the $I = 0$ distribution resulting from $\pi[2s_{1/2}]^{-2}$ removal, which is remarkably similar to the $I = 0$ distribution arising from $\pi[0h_{11/2}]^{-2}$, showing the insensitivity to $\ell$. This is consistent with the analysis of two-nucleon angular correlations (see Section 4, Fig. 4.2), which show large similarities for the angular correlation functions $\Gamma_{jj}^{I=0}(\omega)$ for these two configurations.

Alignment of projectile-like fragments was observed in the fragmentation of $^{46}$Ti [70]. An $19/2^−$ isomeric state in the $−nnp$ residue $^{43}$Sc was shown to be significantly aligned and that the alignment was dependent on the residue momentum. In this case, there may be indirect evaporative contributions that populate the $^{43}$Sc residue and in any case, the direct $−nnp$ reaction is beyond the scope of the current model. Nevertheless, the isomeric ratio is shown to increase and the alignment decrease for the extremes of the residue momentum.

We saw in Section 6.2 that different magnetic substates have rather different heavy residue momentum distributions. The maximally aligned states (classically) involve orbital motion on a plane perpendicular to the $z$-axis (beam direction) and so are expected to have particularly narrow momentum distributions. Here we illustrate this phenomena for the $I = 10$ case described above, shown in Fig. 7.13. Regardless of the residue momentum distribution, the $J_f^\pi = 10^+$ state is expected to be well aligned, with more than more than 50% of events leading to the state with $|\mu| \geq 8$. As seen previously, the central part of the residue momentum distribution is dominated by highly aligned components and if
7.2. HEAVY PROJECTILES

Figure 7.12: $^{206}$Hg Heavy residue momentum distributions following two-proton knockout from $^{208}$Pb($-2p)$. Shown are pure $[0h_{11/2}]^2$ configuration removal for $I = 0$ (solid), 2 (dotted), 4 (dashed), 6 (solid triangles), 8 (open squares) and 10 (solid circles), and pure $[2s_{1/2}]^2$ distribution ($I = 0$, open diamonds). The $I = 0$ $[0h_{11/2}]^2$ and $[2s_{1/2}]^2$ distributions are very similar in shape, particularly compared to the $[0h_{11/2}]^2$ distributions for larger $I$.

An illustration of the subtle effects related to the nucleon pair configuration is given by population of the $^{206}$Hg first and third $3^+$ states. The configurations of these states are predominantly $[2s_{1/2}][1d_{5/2}]$ and $[2s_{1/2}][0g_{7/2}]$ respectively. Critically, both configurations take one nucleon from the $2s_{1/2}$ orbital. Since this orbital has $\ell = 0$, the total orbital angular momenta $L$ must be equal to the $\ell$ of the second orbital (true when one orbital is s-wave). As we saw in Section 6.6 the total orbital angular momentum is critical in determining the residue momentum width. Thus, due to the small configuration mixing, these two $3^+$ states are populated with pure $L = 2$ and $L = 4$ contributions, and the heavy residue momentum distributions, shown in Fig. 7.14 are very different despite the final state spins being identical. The calculations in Fig. 7.14 use the full shell-model two-nucleon amplitudes; we have not assumed that they are populated via the single configurations mentioned earlier. As such, these two states provide a very clear example where the subtle differences discussed in Section 6.6 could be observed.
Figure 7.13: Magnetic substates of the 10+ state in 206Hg populated via $\pi[0h_{11/2}]^2$ removal from 208Pb. Shown are $\mu = 0$ (solid), $|\mu| = 1$ (open circles), 2 (dashed), 3 (open squares), 4 (dotted), 5 (open triangles), 6 (dot-dashed), 7 (open diamonds), 8 (dot-dot-dashed), 9 (solid triangles) and 10 (crosses). The solid circles show the total. States with $|\mu| \geq 8$ are populated with larger than 50% probability, increasing to >90% in the region -100–100 MeV/c (shaded). See also Table 7.12.

| $|\mu|$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|---|---|
| Full  | 2.6 | 5.2 | 5.4 | 5.8 | 6.2 | 6.9 | 7.7 | 8.9 | 10.7 | 14.6 | 26.0 |
| Cut   | 0.2 | 0.4 | 0.4 | 0.5 | 0.8 | 1.1 | 1.9 | 3.5 | 7.4 | 19.2 | 64.6 |

Table 7.12: Fractional populations of different magnetic substates of the heavy residue after $\pi[0h_{11/2}]^2$ removal from 208Pb, populating to the $J\pi_f = 10^+$ final state, based on pure-stripping calculations. Restricting the residue momentum to allow only events within ±100 MeV/c of the central momentum significantly enhances the fractional population of the large $|\mu|$ substates.
Figure 7.14: Residue momentum distributions for two $3^+$ states in $^{206}\text{Hg}$, expected to be populated in two-proton removal from $^{208}\text{Pb}$. The predominant configurations in the two cases are $[2s_{1/2}][1d_{5/2}]$ (solid) and $[2s_{1/2}][0g_{7/2}]$ (dashed), meaning that the states will be populated with total orbital angular momentum $L = 2$ and $L = 4$ respectively. The heavy residue momentum distributions are strikingly different despite the final state being of the same spin. The full-width-half-maximum is 230 MeV/c for the $[2s_{1/2}][1d_{5/2}]$ configuration and 370 MeV/c for $[2s_{1/2}][0g_{7/2}]$ configuration.
Chapter 8

Conclusions

8.1 Summary

The longitudinal momentum distributions of the reaction residue following the sudden removal of a single nucleon from a fast radioactive beam is known to give information on the orbital angular momentum of the removed nucleon [5, 1, 6]. This fact has been exploited by numerous experiments and has provided valuable information on the evolution of nuclear structure in exotic species such as $^{19}$C [93] and $^{32}$Ar [3]. Here, we have developed the formalism for the calculation of residue momentum distributions following the sudden removal of two well-bound nucleons. The width of the momentum distribution is shown to be characteristic of total angular momentum of the nucleon pair $I$ and as such can be used to make final-state spin assignments, particularly for the (usual) case of an even-even projectile where the ground state has spin $J_i = 0$. It is understood that the distribution is essentially a property of the two-nucleon overlap probed at the nuclear surface. The driving factor in changes in the residue momentum distributions as $I$ is varied is shown to be changes in the angular correlations of the two nucleons. The determining factor of the width of the residue momentum distribution is shown to be the total orbital angular momenta $L$ and insight into the relative widths of distributions arising from different nucleon pairs can be gained by considering the relative strengths of the contributing $L$. Observation of the subtle effects associated with different pair combinations will require relatively pure final state configurations. One such example could be observed in the removal of two protons from $^{208}$Pb, where two particular $^{206}$Hg $3^+$ final states are expected to have very different residue momentum distributions due to the dominant configurations involving the $[2s_{1/2}]$ proton orbital.

Increasing the binding energy of the nucleons leads to broader residue momentum distributions, though the separation energy sensitivity is significantly less than that to the total angular momentum. As the projectile mass increases the residue distribution
widths are expected to become more narrow. Both of these observations are a consequence of the uncertainty principle and were made when using simple approximate forms for the radial wave function. If one could study the most peripheral events, by observing the most forward going residues for instance, the residue momentum width is predicted to decrease marginally for high-spin states, but remain essentially constant for $I = 0$.

Recent experimental examples were confronted and excellent agreement for the residue momentum distributions was found for $^{22}\text{Mg}(-2\text{n}),\; ^{28}\text{Mg}(-2\text{p})$ and $^{38}\text{Si}(-2\text{p})$. The final-state inclusive distribution of $^{22}\text{Mg}$ is dominated by the $I = 0$ transition to the $^{20}\text{Mg}$ ground state, and is very narrow as a result. Two-proton removal from $^{28}\text{Mg}$ is the first (and only) example where the cross section and secondary beam intensity were sufficient to extract final-state exclusive momentum distributions, and distributions for both the $0^+$ ground state and $4^+$ excited state the agreement is good, though the distinctive widths are somewhat obscured by the large degree of experimental broadening and still low statistics. Such experiments are now capable of being repeated using higher secondary beam intensities and significantly improved $\gamma$-ray detection efficiencies.

Two-proton removal from $^{208}\text{Pb}$ was considered in order to investigate the difficulties associated with heavy mass projectiles. The primary complication when heavy mass projectiles are concerned is the high density of residue final states below particle separation thresholds. Whereas for examples of two-nucleon knockout for projectiles with $A < 60$, at most four states are populated, removing two protons from $^{208}\text{Pb}$ populates 52 neutron-bound states in $^{206}\text{Hg}$, all relatively strongly. The experimental information in this case is limited to date to the isomeric population ratios for two isomeric states in $^{206}\text{Hg}$. Absence of prompt $\gamma$-ray measurements complicates the calculation of isomeric ratios for states that are known to be fed in $\gamma$-cascades from higher lying states. However, the isomeric ratios for the $5^-$ and $10^+$ isomers in $^{206}\text{Hg}$ are well described when the observed feeding is taken into account for the former. Further, we discussed the associated residue momentum distributions, to which the isomeric ratios for these high-spin states, in principle, show a strong sensitivity. We also highlight the consequent importance of the (experimental) spectrometer momentum transmission in determining the isomeric ratio.

## 8.2 Topics of further interest

### 8.2.1 Heavy systems

Future radioactive beam facilities will provide a wide range of heavy radioactive beams. Here we have described the removal of two protons from $^{208}\text{Pb}$, populating isomeric states in $^{206}\text{Hg}$, offering insight into the possibilities and complications of knockout with heavy projectiles. Though the isomeric ratios are reproduced by current calculations, they are,
by definition, a relative quantity and measurements of the absolute cross sections for individual final states are vital for verification of the theoretical methodology for heavy projectiles. The present experiment did not have adequate monitoring of the incident primary $^{208}$Pb beam intensity to allow the determination of absolute yields.

A simpler test of the reaction mechanism for heavy projectiles would be provided by single-nucleon removal. In particular, single-proton knockout from $^{208}$Pb will populate five (simple) proton hole states, one of which is the $\pi[0h_{11/2}]^{-1}$ isomer with half-life 1.3 s. In addition to the isomeric ratios, the absolute cross sections are needed to quantify deduced spectroscopic factors against e.g. regular kinematics ($e,e'p$) analyses that are available for this stable target [94, 95]. Similarly, one-neutron removal from $^{208}$Pb would populate the $T_{1/2} = 0.8 \text{ s } \nu[0i_{13/2}]^{-1}$ isomeric state, also providing an excellent large orbital angular momentum example. Other proton-removal examples of heavy targets studied with ($e,e'p$) are given in Ref. [95], e.g. $^{90}$Zr, $^{142}$Nd and $^{206}$Pb. These examples will provide a vital benchmark.

### 8.2.2 Isomeric beams

Accurate prediction of isomeric population ratios is clearly important for isomer decay spectroscopy, which relies on the $\gamma$-cascades resulting from direct population of isomeric states. Another topic of interest connected with isomers would be nucleon knockout from secondary beams known to be, at least partially, isomeric. The final states populated could be considerably altered since the initial states would have significantly different configurations. A further consideration would be the need to take into account the effects of the potentially large alignment of the isomeric beam, as were discussed in the $^{208}$Pb($-2p$) example. As an example one might conceive of populating the $^{206}$Hg residue states via two step fragmentation of $^{208}$Pb, populating $^{206}$Hg final states via single-proton knockout from the single-proton knockout residue $^{207}$Tl. We would expect the $T_{1/2} = 1.3 \text{ s } J^\pi = 11/2^-$ isomeric state to be strongly populated, with some excited states in $^{206}$Hg, notably the 10$^+$ isomer only able to be populated via the intermediate $11/2^-$ isomer. Knockout from secondary beams produced via fragmentation reactions, particularly those far from the projectile in mass, will require careful consideration of the secondary beam states in order gain useful spectroscopic information from the secondary reaction.

### 8.2.3 Unlike pair knockout

Recent electron induced knockout experiments have precisely probed unlike pair correlations for $^{12}$C [96]. They find that the unlike nucleon pairs are approximately ten times more prevalent than like-nucleon pairs. It is a question of repeated interest as to whether
8.2. TOPICS OF FURTHER INTEREST

such correlations could be probed via unlike-pair knockout using nuclear targets. Little experimental evidence is available, but the few cross sections available [97, 98] suggest strong enhancement for np knockout over like nucleon pair knockout and significantly more that one would expect from simple combinatorics. We will consider here the systems where np-correlations might be observed via nucleon knockout using a nuclear target.

Firstly, we assume that such correlations could only be studied when the residue final states are populated directly and that it is not possible to distinguish the direct and indirect (evaporative) contributions experimentally. Therefore, we must consider systems with similar proton and neutron separation thresholds, preferably as large as possible. If the neutrons (protons) are weakly bound, we would expect large indirect contributions arising from single proton (neutron) knockout to particle-unbound states. Secondly, we would require that the projectile nucleus in question be relatively low in mass, such that the final residue states cannot be populated via the removal of deeply bound nucleons, either directly or indirectly. These considerations constrain the potential candidates significantly; the best examples will most likely be $^{12}$C and $^{16}$O (see Ref. [39] for like-nucleon stripping from $^{12}$C). This introduces the additional (experimental) complication of distinguishing the $A-2$ residue from the incident beam, since in np removal the two will have the same mass-to-charge ratio. For this reason caution should be placed on current measurements.

Ideally, any experiment would measure single-nucleon and like-pair removal as well as unlike pair removal. The single-nucleon removal results are critical to quantify any potential indirect contributions. If the nucleons are initially well-bound we might expect the single-nucleon removal strength to be largely exhausted below the particle separation thresholds in the $A-1$ daughters. The two-like-nucleon removal results give important verification. Final-state exclusive measurements would be essential to progress such an analysis.

8.2.4 Deformed systems

Recent examples of two-nucleon knockout near the island of inversion observe strongly suppressed experimental cross sections relative to state-of-the-art spherical basis shell model calculations. The interpretation is that large deformation-driving intruder components reduce the overlap of the initial and final states and that the shell model based on a spherical basis is inadequate to track such changes. The introduction of larger model spaces and additional cross-shell excitations can certainly replicate some of the features of deformation within a spherical shell-model basis, but there are severe computational constraints to such calculations. Near-future experiments on exotic nuclei of mass $A>60$ take us beyond the reach of the shell model. Further, deformation must be taken into ac-
count in both structural and reaction dynamical aspects. A new theoretical methodology is required, explicitly incorporating deformation from the outset.

Two-nucleon knockout is particularly useful for investigating changes in structure far from stability. The *island of inversion* defines a region of neutron rich nuclei near \( ^{32}\text{Na} \), dominated by deformation-driving neutron \((sd)^{-2}(fp)^{+2}\) intruder configurations due to quenching of the \( N = 20 \) \( sd - fp \) shell gap by the tensor force. Recent experiments have explored structural changes in the *island of inversion*, manifest as unexpectedly small experimental two-proton knockout cross sections. In two-proton removal from \(^{38}\text{Si}\), discussed in Section 7.1.2, the small cross section was attributed to large mixing of two-particle-two-hole neutron configurations in the residue \( ^{36}\text{Mg} \) wave function. These configurations, inaccessible from the predominantly \( 0\hbar \omega \) \(^{38}\text{Si} \) ground state, are symptomatic of the onset of deformation [12]. Fallon *et al.* consider \(^{32}\text{Mg}(−2p)\), deducing large neutron four-particle-four-hole components in \(^{30}\text{Ne}\), suggesting this increases stability of \(^{29}\text{F} \) and \(^{31}\text{F} \) [11]. A new island of inversion associated with reduction of the \( N = 40 \) shell gap is expected, with \(^{64}\text{Cr} \) predicted to be the most deformed system in the region. Adrich *et al.* observe a strongly suppressed cross section for two-proton removal from \(^{66}\text{Fe} \) (an order of magnitude smaller than found in two-proton removal from \(^{68}\text{Ni} \)), again attributed to deformation-driving intruder configurations [15]. The low-energy first \( 2^+ \) state in \(^{42}\text{Si} \), measured via direct two-proton knockout, shows \(^{42}\text{Si} \) to be a well-deformed oblate rotor and illustrates the disappearance of the spherical \( N = 28 \) shell closure [14]. Such experiments provide major challenges to our quantitative understanding of shell structure changes away from the valley of \( \beta \)-stability.

Whilst the primary interest would be the absolute cross section, which gives indications of structural changes, a signature of deformation in the residue momentum distribution would be of great interest. Calculations for single-nucleon removal for deformed systems are shown to have characteristic features in the momentum distribution that cannot be replicated within a spherical knockout model [33]. In addition to large changes in structure associated with the onset of deformation, such a model would be applicable to cases where both projectile initial and residue final states are known to be deformed. Many secondary beams of deformed projectiles will be available at the next generation of radioactive beam facilities and further theoretical development is required.

### 8.2.5 Weakly populated excited states

An interesting application of like two-nucleon knockout might be removal from nuclei with two protons and two neutrons above \( N = Z \) doubly magic nuclei, e.g. \(^{20}\text{Ne} \) or \(^{44}\text{Ti} \). In such a case we would expect to reasonably describe the structure of the projectile ground state as an inert core with pairs of valence protons and neutrons. The ground state will
be predominately seniority-0, where both pairs of (like) nucleons are coupled to $I = 0$, with small contributions from seniority-4 components where both pairs are coupled to $I \neq 0$. Assuming nucleon removal where there is no dynamic core excitation and cross-shell (core-breaking) interactions, the only way to populate $J_f \neq 0$ seniority-2 states in the final heavy residue is to probe the seniority-4 components of the projectile ground state. Consequently we would expect such transition to be very weak.

In the case of $^{44}$Ti$(-2n)$ we would expect $0^+, 2^+, 4^+$ and $6^+$ states in $^{42}$Ti to be based on simple $^{40}$Ca plus $\pi[0f_{7/2}]^2$ configurations. The $6^+$ state can only be populated via seniority-4 components of the projectile ground state wave function or via dynamic excitation of the two-protons in the interaction. The same is true for the $2^+$ and $4^+$ states, except that they may also be accessed by removing two $sd$-shell protons, giving an insight into relatively simple cross shell interactions. More generally, the study of cross shell excitations is likely to become a key theme of of structure studies at and near shell and subshell closures.

We would expect similar phenomena for any example with only two nucleons in the model space, i.e. two neutron removal from $^{20}$Ne or $^{22}$Mg, or two proton removal from $^{44}$Ti or $^{46}$Ti. The $^{22}$Mg$(-2n)$ reaction was discussed earlier in Section 7.1.1, though in this case, unfortunately, the $4^+$ state is expected to be proton-unbound.
Appendix A

Two-step reactions with thick targets

The low cross sections for populating the most exotic isotopes for nucleon knockout and peripheral fragmentation reactions necessitate the use of thick targets in order to maximise the experimental reaction yield. This opens the possibility of multiple step reactions to populate the reaction residue of interest. In order to accurately compare with experimental production cross sections with such targets, the possible production of the isotope of interest through multi-step reactions must be considered. The LISE++ program [54, 92] can be used to give an estimate of the losses and multi-step contributions, using a method described in Ref. [99]. Here we make a simple estimate based on the integration of the yield through the target. This is of most relevance to the $^{208}$Pb(−2p) example discussed in Section 7.2, where a particularly thick target was used.

In experiments with $^{136}$Xe and $^{197}$Au beams [100], calculations suggested that “secondary reactions in target and degrader do not contribute noticeably to the observed yields of the proton-removal channels”. The thickness of the target used in these experiments were somewhat thinner (2.011 and 1.395 g/cm$^2$ 9Be) than those used in the current RISING experiments (2.525 g/cm$^2$ 9Be, see Section 7.2), though similar proton removal channels were observed.

The following method was provided by Ref. [41]. To be specific, we consider the removal of two protons, where the only indirect two-step process will be sequential events involving the removal of one proton. The fraction of particles that interact in a target $N_x$, where $x$ indicates the number of protons removed, is given by

$$N_x = \frac{0.6}{A_t} \sigma_x T,$$

where $\sigma_x$ is the cross section in units of barns, $T$ is the target thickness in units of g/cm$^2$.
and $A_t$ is the mass number of the target. We wish to compare the production of a particular isotope via one step and two step processes. We divide the target into thin segments of thickness $t$, such that the number of events leading to single proton knockout after the first segment is given by,

$$N_{11} = \frac{0.6}{A_t} \sigma_1 t.$$  \hfill (A.2)

The notation $N_{ab}$ is introduced, with $a$ is the number of protons removed via one-proton knockout events and $b$ is the number of target segments. $\sigma_1$ is the single proton knockout cross section, assumed known from experiment or taken from theory. The fraction of nuclei having undergone two sequential one proton knockouts is zero after the first segment. After two segments the fraction of single-proton removal reactions $N_{12}$, and the fraction of two-step two-proton removal reactions $N_{22}$, are

$$N_{12} = 2 \frac{0.6}{A_t} \sigma_1 t,$$  \hfill (A.3)

$$N_{22} = \left( \frac{0.6}{A_t} \sigma_1 t \right)^2.$$  \hfill (A.4)

If this is extended to $n$ segments where $t = T/n$ we obtain

$$N_{2n} = \left( \frac{0.6}{A_t} \sigma_1 t \right)^2 [1 + 2 + 3 + \ldots + (n - 1)],$$

$$= \left( \frac{0.6}{A_t} \sigma_1 \frac{T}{n} \right)^2 \left( \frac{n(n - 1)}{2} \right),$$

$$= \frac{1}{2} \left( \frac{0.6}{A_t} \sigma_1 T \right)^2,$$  \hfill (A.5)

where for the last line we have taken the limit as $n \to \infty$ (i.e. infinitely thin segment $t$).

The number of events from single-event two-nucleon knockout is given by Eq. A.1, giving the ratio of sequential to direct events as

$$\frac{N_{2n}}{N_2} = \frac{0.6}{2A_t} \sigma_1 T \left( \frac{\sigma_1}{\sigma_2} \right).$$  \hfill (A.6)

Note that some assumptions have been made. Firstly, that the cross section for one-nucleon knockout is assumed to be independent of the mass number i.e. $\sigma(A_t \rightarrow A_t - \Delta A) \approx \sigma(A \rightarrow A - \Delta A)$. This assumption was also made in Ref. [99]. Secondly, no beam attenuation within the target is taken into account and the cross sections are assumed to not vary with energy.
Appendix B

Peer-reviewed publications

The work of this thesis has resulted in four publications. Papers 1, 3 and 4 were published in Physical Review C and paper 2 was published in Physical Review Letters.

1. One- and Two-neutron removal from neutron-rich carbon isotopes
   E. C. Simpson and J. A. Tostevin,
   Physical Review C 79, 024616 (2009)

2. Two-Nucleon Knockout Spectroscopy at the Limits of Nuclear Stability
   E. C. Simpson, J. A. Tostevin, B. A. Brown, D. Bazin and A. Gade,
   Physical Review Letters 102, 132502 (2009)

3. Longitudinal momentum distributions of the reaction residues following fast two-nucleon knockout reactions
   E. C. Simpson, J. A. Tostevin, D. Bazin and A. Gade,
   Physical Review C 79, 064621 (2009)

4. Population of low-seniority isomeric states of $^{206}$Hg by two-proton knockout reactions at relativistic energies
   E. C. Simpson, J. A. Tostevin, Zs. Podolyák, P. H. Regan and S. J. Steer,
   Physical Review C 80, 064608 (2009)

Note: the published versions of the above papers have been removed. They are available on request.
B.1 One- and Two-neutron removal from neutron-rich carbon isotopes

The following article was published in Physical Review C on 25th February 2009.
B.2 Two-Nucleon Knockout Spectroscopy at the Limits of Nuclear Stability

The following article was published in Physical Review C on 3rd April 2009.
B.3 Longitudinal momentum distributions of the reaction residues following fast two-nucleon knockout reactions

The following article was published in Physical Review C on 29\textsuperscript{th} June 2009.
B.4 Population of low-seniority isomeric states of $^{206}$Hg by two-proton knockout reactions at relativistic energies

The following article was published in Physical Review C on 8th December 2009.
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